Selfadhesivity in Gaussian conditional independence structures

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Adhesive extensions of entropy vectors

- Let $(\xi_i : i \in N)$ be a vector of jointly distributed discrete random variables.
- We associate its entropy vector $h_\xi(I) := H(\xi_I)$ where $\xi_I = (\xi_i : i \in I)$ is a marginal distribution and $H(\xi) := -\sum_x p_\xi(x) \log p_\xi(x)$ is the Shannon entropy.
- The vectors h_ξ are polymatroids and satisfy many, many other properties ...

Theorem (Matúš (2007))

Every pair of random vectors $(\xi_i : i \in N)$ and $(\eta_i : i \in M)$ with $L = N \cap M$ has an adhesive extension over L, i.e., there exists a random vector $(\zeta_i : i \in NM)$ such that:

- $\zeta_N = \xi$ *and* $\zeta_M = \eta$,
- $[\zeta_N \perp\!\!\!\perp \zeta_M \mid \zeta_L]$.

Selfadhesivity

- Fix a discrete random vector ξ and a subset $L \subseteq N$.
- Choose a set M with $|M| = |N|$ and a bijection $\pi : N \to M$ which fixes L pointwise. The pair (M, π) is an L-copy of N.
- ξ is selfadhesive at L if there exists an adhesive extension ζ of ξ and $\pi(\xi)$.
- ξ is selfadhesive if it is selfadhesive at every L.

Corollary (Matúš (2007))

Entropy vectors are selfadhesive.

Refresher on Gaussian conditional independence

- A joint (regular!) Gaussian distribution of N random variables $(\xi_i : i \in N)$ is specified by its mean vector $\mu \in \mathbb{R}^N$ and its covariance matrix $\Sigma \in \mathrm{PD}_N$.
- The marginal ξ_I has covariance $\Sigma_I \coloneqq \Sigma_{I,I}$, which is a principal submatrix.
- The conditional $\xi_I \mid \xi_{I^c} = x$ has covariance $\Sigma^{I^c} \coloneqq \Sigma_I - \Sigma_{I,I^c} \Sigma_{I^c}^{-1} \Sigma_{I^c,I}$, which is the Schur complement of I^c in Σ.
- Marginal independence $[\xi_I \perp\!\!\!\perp \xi_J]$ is equivalent to $\Sigma_{I,J} = 0$.
- Conditional independence $[\xi_I \perp\!\!\!\perp \xi_J \mid \xi_K]$ is equivalent to $\operatorname{rk} \Sigma_{IK,JK} = |K|$.

Properties of CI structures

- The set $[\![\Sigma]\!] \coloneqq \{ [\xi_I \perp\!\!\!\perp \xi_J \mid \xi_K] : \operatorname{rk} \Sigma_{IK,JK} = |K| \}$ is the CI structure of the Gaussian distribution(s) ξ with covariance matrix Σ.
- A property of CI structures is a vector $\mathfrak{p}$ which assigns to every finite set N a set of CI structures.
  - The semigraphoids $\mathfrak{sg}(N)$ on N form a property.
  - $\mathfrak{g}^+(N) \coloneqq \{ [\![\Sigma]\!] : \Sigma \in \mathrm{PD}_N \}$ form the property of being Gaussian.

Properties are ordered by inclusion: $\mathfrak{g}^+ \leq \mathfrak{sg}$.

"Every Gaussian distribution induces a semigraphoid."

Adhesive extensions of covariance matrices

Theorem

Every pair of positive definite matrices $\Sigma \in \mathrm{PD}_N$ *and* $\Delta \in \mathrm{PD}_M$ *with* $L = N \cap M$ *has an adhesive extension over* L*, i.e., there exists a* $\Phi \in \mathrm{PD}_{NM}$ *such that:*

- $\Phi_N = \Sigma$ *and* $\Phi_M = \Delta$,
- $[N \perp\!\!\!\perp M \mid L] \in [\![\Phi]\!]$.

Proof.

Easy exercise in linear algebra! □

Corollary

The multiinformation vectors of regular Gaussian distributions are selfadhesive.

Structural selfadhesivity

If all Gaussians have some property $\mathfrak{p}$ **and** are selfadhesive, then this reveals more structure than $\mathfrak{p}$ alone.

Let $\mathfrak{p}$ be a property of CI structures. Its selfadhesion is the property $\mathfrak{p}^{\mathsf{sa}}$ defined as follows. $\mathcal{L} \in \mathfrak{p}^{\mathsf{sa}}(N)$ if and only if for every $L \subseteq N$ and an L-copy (M, π) of N there exists $\overline{\mathcal{L}} \in \mathfrak{p}(NM)$ with:

- $\overline{\mathcal{L}}|_N = \mathcal{L}$ and $\overline{\mathcal{L}}|_M = \pi(\mathcal{L})$,
- $[N \perp\!\!\!\perp M \mid L] \in \overline{\mathcal{L}}$.

Lemma

$\mathfrak{g}^+$ is a fixed point of $\cdot^{\mathsf{sa}}$. Hence if $\mathfrak{g}^+ \leq \mathfrak{p}$, then also $\mathfrak{g}^+ \leq \mathfrak{p}^{\mathsf{sa}} \leq \mathfrak{p}$.

Computations

Apply selfadhesion to strengthen known necessary properties of Gaussianity:

- Structural semigraphoids $\mathfrak{sg}_*$: Studený (1994):

Computation

There are $508\,817$ *gaussoids on* $n = 5$ *random variables modulo isomorphy. Of these* $336\,838$ *satisfy* $\mathfrak{sg}_*$ *and* $335\,047$ *of them satisfy* $\mathfrak{sg}_*^{\mathsf{sa}}$.

- Orientable gaussoids $\mathfrak{o}$: Boege–D'Alì–Kahle–Sturmfels (2019):

Computation

All orientable gaussoids on $n = 4$ *are Gaussian. Precisely* $175\,215$ *gaussoids satisfy* $\mathfrak{o}$ *and* $168\,010$ *satisfy* $\mathfrak{o}^{\mathsf{sa}}$.

Computations

Computation

The properties $\mathfrak{sg}_* \wedge \mathfrak{o}$ *and* $\mathfrak{sg}_*^{\mathsf{sa}} \wedge \mathfrak{o}$ *coincide at* $n = 5$ *with* $175\,139$ *isomorphy types. On the other hand,* $\mathfrak{sg}_* \wedge \mathfrak{o}^{\mathsf{sa}}$, $\mathfrak{sg}_*^{\mathsf{sa}} \wedge \mathfrak{o}^{\mathsf{sa}}$ *and* $(\mathfrak{sg}_* \wedge \mathfrak{o})^{\mathsf{sa}}$ *coincide at* $n = 5$ *with* $167\,989$ *types.*

This yields new inference rules for Gaussian conditional independence:

$$[i \perp\!\!\!\perp j \mid km] \wedge [i \perp\!\!\!\perp m \mid l] \wedge [j \perp\!\!\!\perp k \mid i] \wedge [j \perp\!\!\!\perp m] \wedge [k \perp\!\!\!\perp l] \;\Rightarrow\; [i \perp\!\!\!\perp j].$$

→ https://mathrepo.mis.mpg.de/SelfadhesiveGaussianCI/ ←

References

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