The Ingleton inequality for random variables

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Matroids

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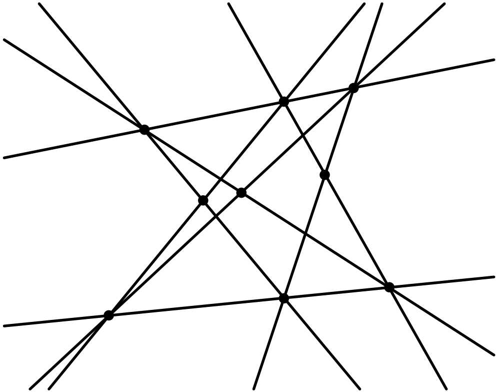

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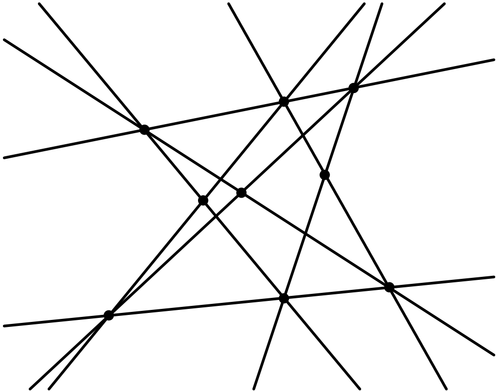

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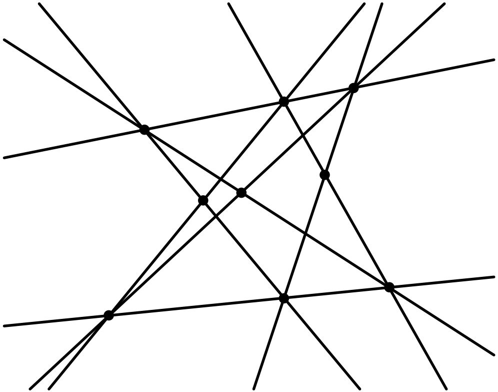
- Matroids are combinatorial structures which model “special position” relations in geometry.
  - For example the matroid of a set of points in the projective plane records which triples of points lie on a line.
- Non-realizability of matroids captures the (non-obvious) laws of projective geometry.



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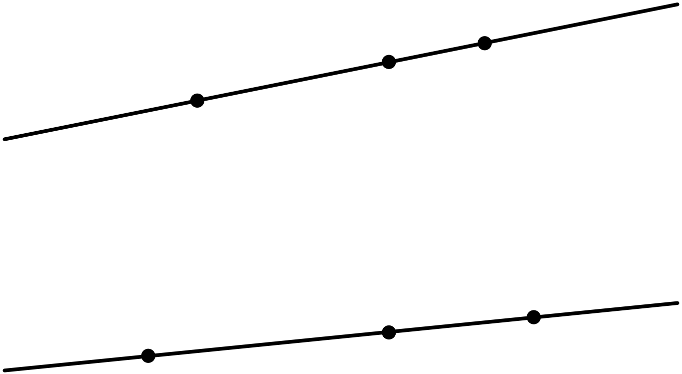

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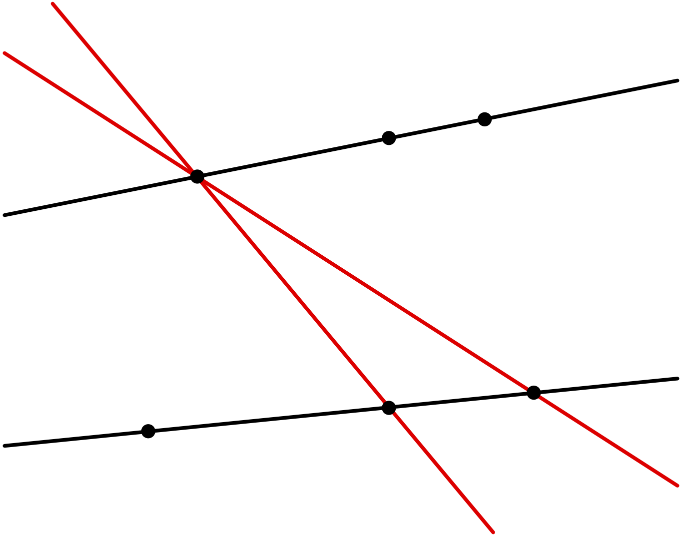

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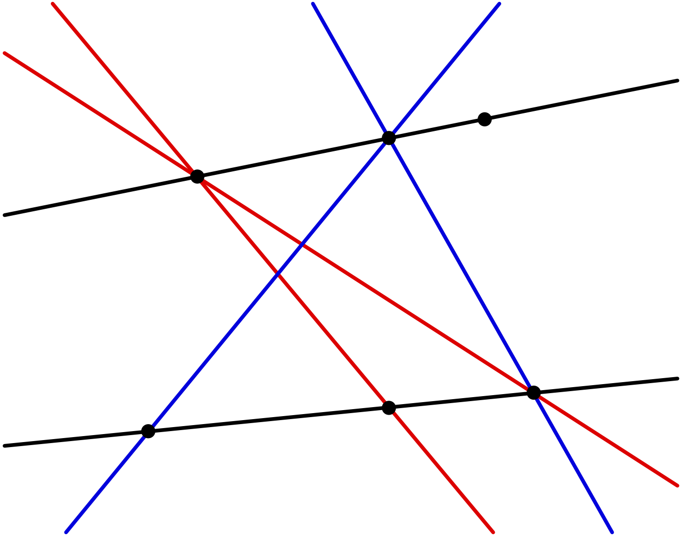

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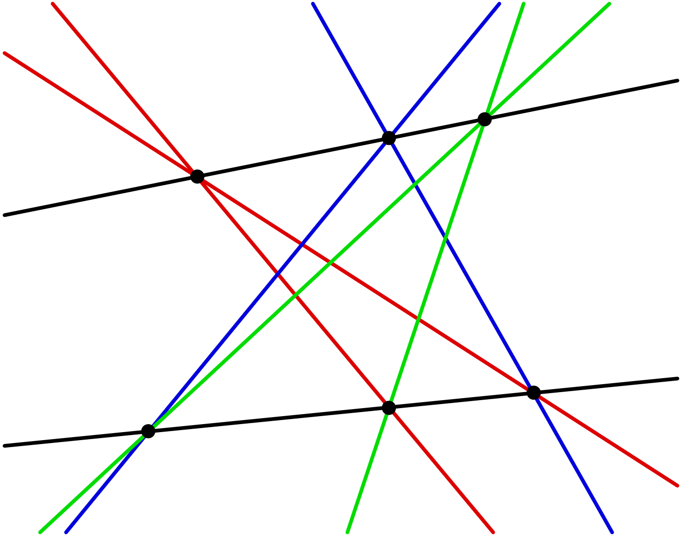

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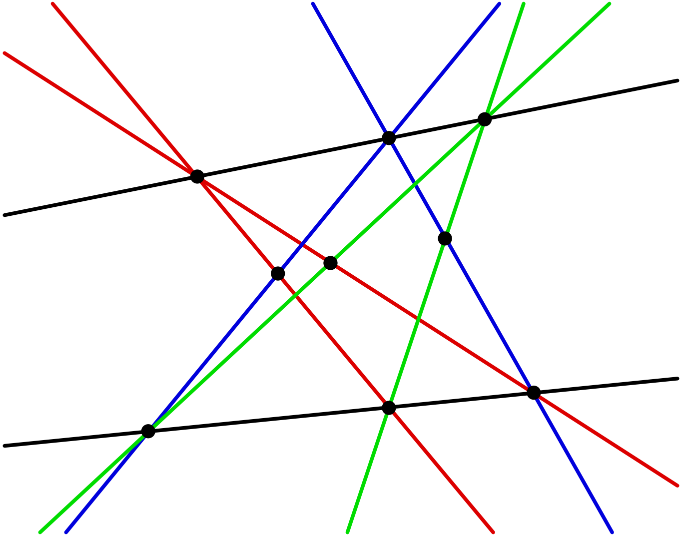

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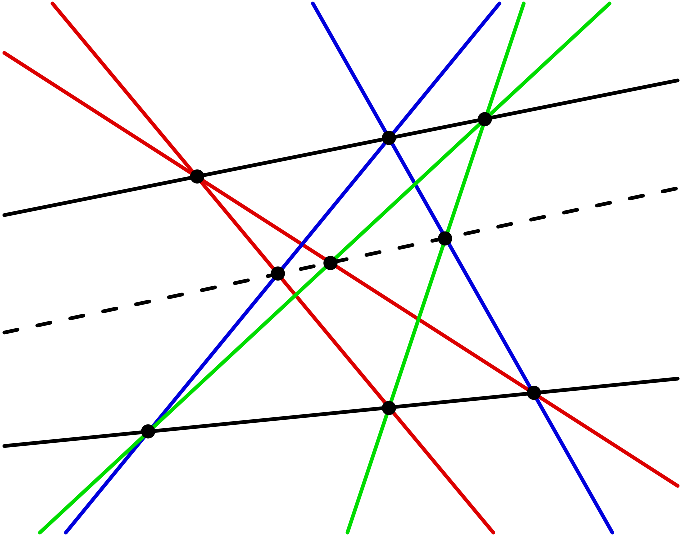

Entropy

Let X be a random variable taking finitely many values $\{1, \ldots, d\}$ with positive probabilities. Its *Shannon entropy* is

$$H(X) := \sum_{i=1}^{d} p(X = i) \log {}^{1}/_{p(X = i)}.$$

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- H is continuous on $\Delta(d)$ and analytic on the interior.
- A random vector $X \in \Delta(d_1, \ldots, d_n)$ is a random variable in $\Delta(\prod_{i=1}^{n} d_i)$, so the definition of H extends to vectors.
- The random vector $X = (X_1, \ldots, X_n)$ has 2^n marginal random vectors and we collect their entropies in an entropy profile $h_X : 2^{[n]} \to \mathbb{R}$.
  - For example (X, Y) has entropy profile $(0, H(X), H(Y), H(X, Y)) \in \mathbb{R}^4$.

Entropy as information

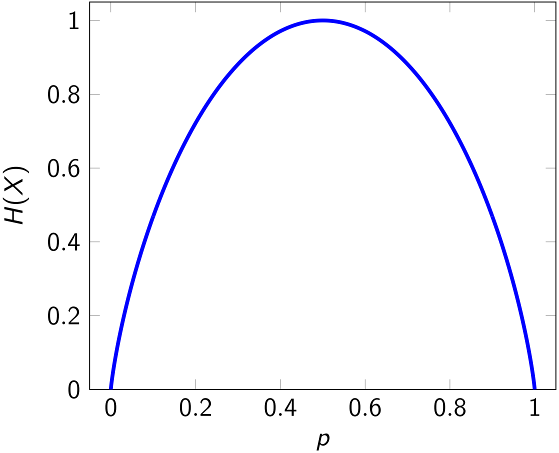

Figure: Entropy of a binary random variable X as a function of $p = p(X = \text{heads})$.

The entropy region and information inequalities

Let $\mathbf{H}_n^* \subseteq \mathbb{R}^{2^n}$ consist of all h_X where X is an n-variate discrete random vector. $\mathbf{H}_n^*$ is the image of $\bigcup_{d_1=1}^{\infty} \cdots \bigcup_{d_n=1}^{\infty} \Delta(d_1, \ldots, d_n)$ under the transcendental map $X \mapsto h_X$.

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- Elements of the dual cone (linear information inequalities) can give bounds for optimization problems.

Shannon inequalities

- A function $h : 2^N \to \mathbb{R}$ is a polymatroid if
  - $h(\emptyset) = 0$,
  - $h(I \mid K) := h(IK) - h(K) \geq 0$ for disjoint I and K,
  - $h(I : J \mid K) := h(IK) + h(JK) - h(IJK) - h(K) \geq 0$ for disjoint I, J, K.

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- The set $\mathbf{P}_N$ of polymatroids is a polyhedral cone in $\mathbb{R}^{2^N}$ and $\mathbf{P}_N \supseteq \overline{\mathbf{H}_N^*} \to$ ITIP.
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Theorem ([Mat07])

$\overline{\mathbf{H}_N^*}$ *is not polyhedral for* $|N| \geq 4$.

- GMM conjecture: $\overline{\mathbf{H}_N^*}$ is not semialgebraic for $|N| \geq 4$.

Information inequalities abound

Rule [43] Given:

$$
\begin{aligned}
& aI(A;B) \\
\leq\ & bI(A;B|C) + cI(A;C|B) + zI(B;C|A) \\
+\ & eI(A;B|D) + fI(A;D|B) \\
+\ & (b'+d'+z)I(B;D|A) + hI(C;D) \\
+\ & iI(C;D|A) + zI(C;D|B)
\end{aligned}
$$

and

$$
\begin{aligned}
& a'I(A;B) \\
\leq\ & b'I(A;B|C) + c'I(A;C|B) + d'I(B;C|A) \\
+\ & e'I(A;B|D) + f'I(A;D|B) + g'I(B;D|A) \\
+\ & h'I(C;D) + i'I(C;D|A) + j'I(C;D|B)
\end{aligned}
$$

Get:

$$
\begin{aligned}
& (a+a'+z)I(A;B) \\
\leq\ & (a+b+c+f+b'+2z)I(A;B|C) \\
+\ & (-a+b+c+e+c'+z)I(A;C|B) \\
+\ & (d'+z)I(B;C|A) + (e+e'+z)I(A;B|D) \\
+\ & (f+f')I(A;D|B) \\
+\ & (-a'+b'+e'+g'+i')I(B;D|A) \\
+\ & (h+h'+z)I(C;D) + (i+i')I(C;D|A) \\
+\ & (j')I(C;D|B)
\end{aligned}
$$

Using: RS is copy of CD over AB
Substitutions: $A\ C\ R$ S; $AD\ B\ R\ S$

Abbreviated Proof of (75): T: D-copy of A over BCRS.
L1: -a.c. +c.d. +r.cd.a +c.s.a +b.d.s +a.bs.d +2a.cr.bs +a.bs.cr +d.r.abcs +d.s.abcr
SL1: d.t.a +c.d.t +a.t.cd +c.r.t +a.t.cr +d.r.act +b.t.acdr +a.t.bs +c.s.at +b.t.acs +d.t.s +a.s.dt +b.d.ast +c.t.abds +a.r.bcst +r.ad.bcst +s.ad.bcrt +d.t.abcrs C2L1: 3t.ad.bcrs
S: C-copy of A over BDR.
L2: -2a.c. +2c.d. +a.b.cr +2a.c.br +c.ar.b +a.b.dr +4a.d.br +2a.br.d +2d.br.a +2r.cd.a +d.r.abc
SL2: c.s.b +a.b.cs +c.d.s +a.s.cd +d.s.abc +3a.s.br +3c.s.br +c.r.abs +d.r.s +a.s.dr +d.r.abs +d.br.as +c.r.ads +b.s.acdr +2c.s.abdr +2d.s.abcr
C2L2: 7s.ac.bdr
R: D-copy of C over AB.
S: c.r.a +3c.r.b +d.r.a +7d.r.b +c.d.r +2b.r.acd +r.ab.cd +9c.r.abd +3d.r.abc
C2: 16r.cd.ab

Randall Dougherty, Chris Freiling, and Kenneth Zeger. *Non-Shannon Information Inequalities in Four Random Variables*. 2011. arXiv: 1104.3602v1 [cs.IT]

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All of these are **linear** in h. Even though entropy is a transcendental function, many of these conditions are **polynomial** in the probabilities $\rightarrow$ algebraic statistics.

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These are conditional linear information inequalities and they can sometimes tell apart honest boundary parts of $\mathbf{H}_n^*$ from fake boundary parts on $\overline{\mathbf{H}_n^*}$.

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- $\mathbf{J}_4$ is simplicial and its facets are induced by conditional independence functionals and the Ingleton functional.

Conditional Ingleton inequalities

Theorem ([KR13] & [Stu21] & [Boe23])

Up to symmetry there are precisely ten minimal sets of conditional independence assumptions on four random variables which ensure $I \geq 0$.

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Which of these laws holds on $\overline{\mathbf{H}_4^*}$? (Some do, some don't ...)

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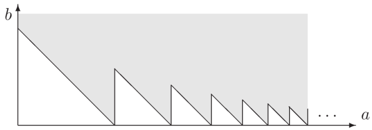

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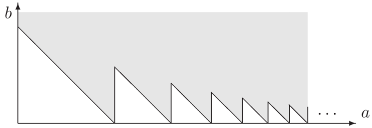

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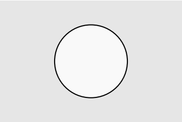

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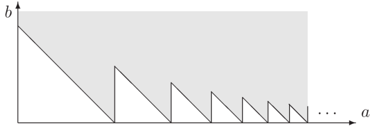
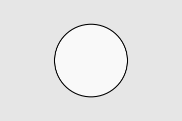
Problem

Find a description of the boundary of $\mathbf{H}_3^*$.





Thank you!

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