# Computational problems in probabilistic reasoning

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based on joint work with
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SIAM AG 2023, TU Eindhoven,
Computational real algebraic geometry III,
13 July 2023

# Matroids and laws of geometry

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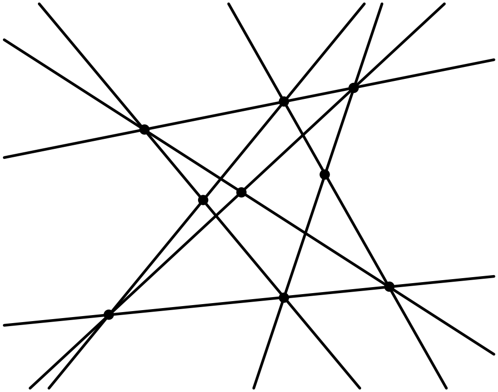

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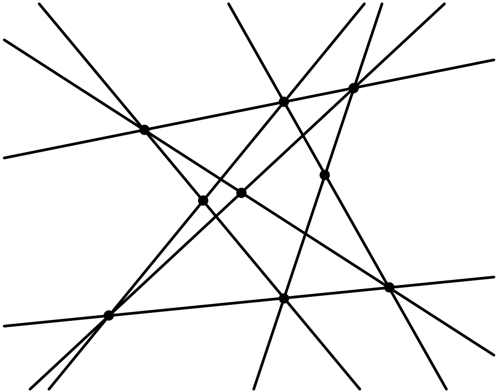

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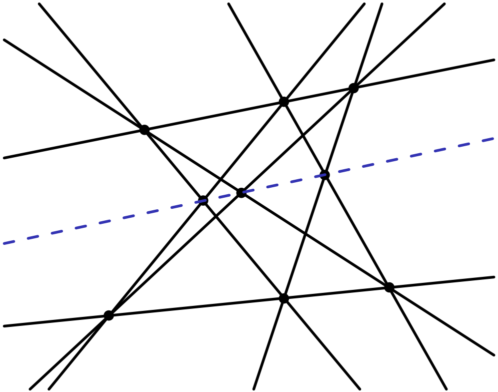
  - For example the matroid of a set of points in the projective plane records which triples of points lie on a line.
- Non-realizability of matroids captures the (non-obvious) laws of geometry.



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**Laws of probabilistic reasoning**

*Let $X_1, \dots, X_n$ be jointly distributed random variables. Assume that $X_i \perp\!\!\!\perp X_j \mid X_K$ for some choices of $i, j \in [n]$ and $K \subseteq [n] \setminus \{i, j\}$. Which other CI statements $X_r \perp\!\!\!\perp X_s \mid X_T$ also hold?*

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Even though entropy is a transcendental function, all of these conditions are **polynomial** in the probabilities → algebraic statistics.

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- The general CI statement $X_i \perp\!\!\!\perp X_j \mid X_K$ requires this decomposition for all $2^{|K|}$ slices of the marginal tensor $p^{\{i,j\} \cup K} \to$ **many** quadratic equations.

# Models and axioms

To every formula $\varphi = \bigwedge_p [X_{i_p} \perp\!\!\!\perp X_{j_p} \mid X_{K_p}] \Rightarrow \bigvee_q [X_{r_q} \perp\!\!\!\perp X_{s_q} \mid X_{T_q}]$ there is a semialgebraic set $K(\varphi)$ of counterexamples, i.e., real $2 \times 2 \times 2 \times 2$ tensors:

- $(\mathscr{P})$ with non-negative entries,
- $(\mathscr{I})$ satisfying all $X_{i_p} \perp\!\!\!\perp X_{j_p} \mid X_{K_p}$ but
- $(\mathscr{M})$ satisfying none of the $X_{r_q} \perp\!\!\!\perp X_{s_q} \mid X_{T_q}$.

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## Conjecture

*The problem of deciding validity for binary distributions is $\forall\mathbb{R}$-complete. Moreover, all real algebraic numbers are necessary to certify invalidity.*

But in $n = 4$ we expect every model to be rationally realizable.

# Known laws I

**Theorem ([Mat18])**

*The following laws are valid and complete for 3 binary random variables* :*

$$[X \perp\!\!\!\perp Y] \wedge [X \perp\!\!\!\perp Z \mid Y] \Rightarrow [X \perp\!\!\!\perp Y \mid Z] \wedge [X \perp\!\!\!\perp Z] \tag{M1}$$

$$[X \perp\!\!\!\perp Y \mid Z] \wedge [X \perp\!\!\!\perp Z \mid Y] \Rightarrow [X \perp\!\!\!\perp Y] \wedge [X \perp\!\!\!\perp Z] \tag{M2}$$

$$[X \perp\!\!\!\perp Y] \wedge [X \perp\!\!\!\perp Y \mid Z] \Rightarrow [X \perp\!\!\!\perp Z] \vee [Y \perp\!\!\!\perp Z]. \tag{M3}$$

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- SAT solvers can be used to derive more axioms logically implied by those above, to count or enumerate structures satisfying these axioms.

# Known laws II

**Theorem ([Šim07]*)**

*The following laws are valid for 4 binary random variables:*

$$[X \perp\!\!\!\perp Y \mid Z, W] \wedge [X \perp\!\!\!\perp Y \mid Z] \wedge [X \perp\!\!\!\perp W] \wedge [Z \perp\!\!\!\perp W] \;\Rightarrow\; [X \perp\!\!\!\perp W \mid Z] \vee [Y \perp\!\!\!\perp W] \tag{Š1}$$

$$[X \perp\!\!\!\perp Y \mid Z, W] \wedge [X \perp\!\!\!\perp Y \mid Z] \wedge [X \perp\!\!\!\perp W] \wedge [Y \perp\!\!\!\perp Z] \;\Rightarrow\; [X \perp\!\!\!\perp W \mid Y] \vee [Z \perp\!\!\!\perp W] \tag{Š2}$$

$$[X \perp\!\!\!\perp Y \mid Z, W] \wedge [X \perp\!\!\!\perp Y] \wedge [Y \perp\!\!\!\perp Z] \wedge [Z \perp\!\!\!\perp W] \wedge [X \perp\!\!\!\perp W] \;\Rightarrow\; [X \perp\!\!\!\perp W \mid Z] \vee [Z \perp\!\!\!\perp W \mid Y] \tag{Š3}$$

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$$[X \perp\!\!\!\perp Y \mid Z, W] \wedge [X \perp\!\!\!\perp Y] \wedge [Y \perp\!\!\!\perp Z] \wedge [Y \perp\!\!\!\perp W] \wedge [X \perp\!\!\!\perp Y \mid Z] \wedge [Y \perp\!\!\!\perp Z \mid X] \;\Rightarrow\; [X \perp\!\!\!\perp W \mid Z] \vee [X \perp\!\!\!\perp Y \mid W]. \tag{Š5}$$

* (Š3) was incorrect in [Šim07].

# Audience participation

What is the vanishing ideal of the set of real non-negative $2 \times 2 \times 2 \times 2$ tensors $p$ which satisfy

$$\left.\begin{array}{ll} p_{0000}p_{1100} = p_{0100}p_{1000} & p_{0001}p_{1101} = p_{0101}p_{1001} \\ p_{0010}p_{1110} = p_{0110}p_{1010} & p_{0011}p_{1111} = p_{0111}p_{1011} \end{array}\right\} \quad [X \perp\!\!\!\perp Y \mid Z, W]$$

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Open e.g. $[X \perp\!\!\!\perp Y \mid Z, W] \wedge [X \perp\!\!\!\perp Z \mid Y] \wedge [X \perp\!\!\!\perp W \mid Y] \;\Rightarrow\; ?$

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These certificates are not used in practice. Why?

# References


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