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The Gaussian CI inference problem

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**MATHEMATISCHE
KOMPLEXITÄTSREDUKTION**

Gaussian conditional independence

Consider random variables $(\xi_i)_{i \in N} \sim \mathcal{N}(\mu, \Sigma)$. The *conditional independence (CI) statement* $\xi_i \perp\!\!\!\perp \xi_j \mid \xi_K$ conveys, informally, that if ξ_K is known, then learning the value of ξ_i does not give any information about ξ_j .

Definition

The polynomial $\Sigma[K] := \det \Sigma_{K,K}$ is a *principal minor* of Σ and $\Sigma[ij|K] := \det \Sigma_{iK,jK}$ is an *almost-principal minor*.

If Σ is positive-definite, then $\Sigma[K] > 0$, and $\xi_i \perp\!\!\!\perp \xi_j \mid \xi_K$ holds if and only if $\Sigma[ij|K] = 0$.



Almost-principal minors

$$\Sigma[ij] = x_{ij}$$

$$\Sigma[ij|k] = x_{ij}x_{kk} - x_{ik}x_{jk}$$

$$\Sigma[ij|kl] = x_{ij}x_{kk}x_{ll} - x_{il}x_{jl}x_{kk} + x_{il}x_{jk}x_{kl} + x_{ik}x_{jl}x_{kl} - x_{ij}x_{kl}^2 - x_{ik}x_{jk}x_{ll}$$

$$\begin{aligned}\Sigma[ij|klm] = & x_{ij}x_{kk}x_{ll}x_{mm} + x_{im}x_{jm}x_{kl}^2 - x_{im}x_{jl}x_{kl}x_{km} - x_{il}x_{jm}x_{kl}x_{km} + x_{il}x_{jl}x_{km}^2 \\ & - x_{im}x_{jm}x_{kk}x_{ll} + x_{im}x_{jk}x_{km}x_{ll} + x_{ik}x_{jm}x_{km}x_{ll} - x_{ij}x_{km}^2x_{ll} \\ & + x_{im}x_{jl}x_{kk}x_{lm} + x_{il}x_{jm}x_{kk}x_{lm} - x_{im}x_{jk}x_{kl}x_{lm} - x_{ik}x_{jm}x_{kl}x_{lm} \\ & - x_{il}x_{jk}x_{km}x_{lm} - x_{ik}x_{jl}x_{km}x_{lm} + 2x_{ij}x_{kl}x_{km}x_{lm} + x_{ik}x_{jk}x_{lm}^2 \\ & - x_{ij}x_{kk}x_{lm}^2 - x_{il}x_{jl}x_{kk}x_{mm} + x_{il}x_{jk}x_{kl}x_{mm} + x_{ik}x_{jl}x_{kl}x_{mm} \\ & - x_{ij}x_{kl}^2x_{mm} - x_{ik}x_{jk}x_{ll}x_{mm}\end{aligned}$$

⋮



Gaussian CI models

Definition

A *CI constraint* is a CI statement $\xi_i \perp\!\!\!\perp \xi_j \mid \xi_K$ or its negation $\neg(\xi_i \perp\!\!\!\perp \xi_j \mid \xi_K)$. They are *algebraic conditions* on the entries of Σ , equivalent to vanishing or non-vanishing of the almost-principal minors $\Sigma[ij|K]$.

Definition

The *model* of a set of CI constraints is the set of all positive-definite matrices which satisfy the constraints.



Gaussian CI models

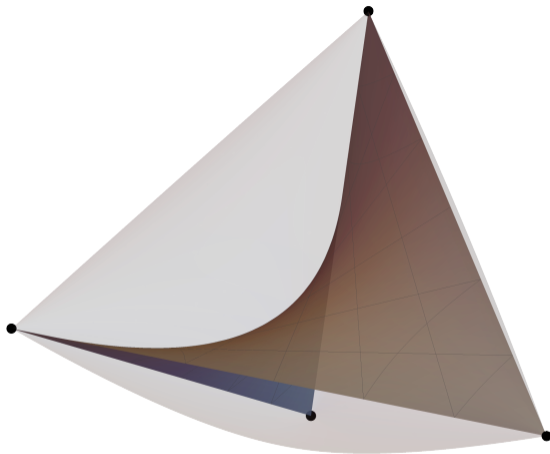


Figure: Gaussian model $\Sigma[12|3] = 0$ inside the elliptope.



Models and inference

Consider two sets of CI statements \mathcal{P} and \mathcal{Q} :

$$\bigwedge \mathcal{P} \Rightarrow \bigvee \mathcal{Q}$$



Models and inference

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$$\begin{array}{ccc} \bigwedge \mathcal{P} \Rightarrow \bigvee \mathcal{Q} & \iff & \mathcal{P} \cup \neg \mathcal{Q} \\ \text{is not valid} & & \text{has a model} \end{array}$$



Models and inference

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Reasoning about relevance statements in normally distributed random variables is **the same** as reasoning about the vanishing of very special kinds of determinants on very special kinds of varieties inside the positive-definite matrices.



Examples of CI inference

Consider a general positive-definite 3×3 correlation matrix

$$\Sigma = \begin{pmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{pmatrix}.$$

- If $\Sigma[12|3] = a - bc$ and $\Sigma[13|] = b$ vanish, then $\Sigma[12|] = a$ and $\Sigma[13|2] = b - ac$ must vanish as well:

$$(12|3) \wedge (13|) \Rightarrow (12|),$$

$$(12|3) \wedge (13|) \Rightarrow (13|2).$$



Examples of CI inference

$$\Sigma = \begin{pmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{pmatrix}$$

- If $\Sigma[12|] = a$ and $\Sigma[12|3] = a - bc$ vanish, then $bc = \Sigma[13|] \cdot \Sigma[23|]$ must vanish:

$$(12|) \wedge (12|3) \Rightarrow (13|) \vee (23|).$$



No finite set of axioms

















































“There is no finite complete axiomatization of Gaussian CI”:















































Theorem (Sullivant 2009)

As the matrix size n grows, there exist valid inference rules for Gaussians which need arbitrarily many antecedents.

$$\begin{aligned} (12|3) \wedge (23|4) \wedge (34|1) \wedge (14|2) &\Rightarrow (12|) && (n = 4) \\ (12|3) \wedge (23|4) \wedge (34|5) \wedge (45|1) \wedge (15|2) &\Rightarrow (12|) && (n = 5) \\ (12|3) \wedge (23|4) \wedge (34|5) \wedge (45|6) \wedge (56|1) \wedge (16|2) &\Rightarrow (12|) && (n = 6) \\ &&& \vdots \end{aligned}$$



 Model M1 4 0 2 -2 0 4 1 -3 2 1 4 -3 -2 -3 -4	 Model M2 4 1 -3 -2 1 4 -3 2 -3 4 -3 2 2 -3 -4	 Model M3 4 -3 -3 -3 -3 4 3 2 -3 3 4 1 -3 2 1 4	 Model M4 4 0 1 -1 0 4 1 1 1 1 4 1 -1 1 1 4	 Model M5 4 0 2 -2 0 4 2 -3 2 2 4 -3 -2 -3 -4	 Model M6 4 0 3 -2 0 4 -2 -3 3 2 4 -3 -2 -3 -4
 Model M7 4 0 -2 -2 0 4 -3 1 -2 -3 4 -1 2 1 -1 4	 Model M8 4 1 -2 -2 1 4 -3 2 -2 -3 4 -1 2 2 -1 4	 Model M9 4 2 -3 3 2 4 -3 1 -3 -3 4 -2 3 1 -2 4	 Model M10 4 1 -2 -2 1 4 -3 2 -2 -3 4 -1 2 2 -2 4	 Model M11 4 0 -3 -3 0 4 1 2 -3 1 4 2 3 2 2 4	 Model M12 4 0 -2 -1 0 4 -3 -2 -2 -3 4 0 1 -2 0 4
 Model M13 4 1 1 2 -2 -1 4 -2 -2 2 -4 -3 -2 -2 2 -3 4	 Model M14 4 1 1 2 -2 1 4 -1 -2 -2 1 4 2 -2 -2 2 4	 Model M15 6 1 4 -2 1 6 -5 3 -4 -5 6 -3 2 3 -3 6	 Model M16 4 0 2 -1 0 4 -2 -3 2 2 4 -3 -1 -3 -3 4	 Model M17 4 0 -2 1 0 4 -3 2 -2 -3 4 -2 1 2 -2 4	 Model M18 4 0 -2 2 0 4 -3 -1 4 0 4 -1 2 -1 -1 4
 Model M19 4 0 2 -2 0 4 -1 -3 2 -1 4 0 -2 -3 0 4	 Model M20 4 0 -2 -3 0 4 2 -1 -2 2 4 -1 -3 -1 1 4	 Model M21 4 0 -2 2 0 4 -2 -2 -2 2 4 -1 2 -2 -1 4	 Model M22 4 0 -2 2 0 4 -2 -2 -2 2 4 -1 2 2 -1 4	 Model M23 4 0 2 -1 0 4 -1 2 2 1 4 2 -1 -2 2 4	 Model M24 6 0 -3 1 0 6 -4 3 -3 4 -6 2 1 3 -2 6
 Model M25 8 0 4 -3 0 8 4 -7 4 4 8 -6 -3 -7 -6 8	 Model M26 8 -3 -6 -6 -3 8 4 4 -6 4 8 2 -6 4 2 8	 Model M27 8 0 4 -2 0 8 2 -5 4 2 8 4 -2 -5 -8 8	 Model M28 10 4 -5 -8 -4 10 2 5 -5 2 10 1 -8 5 11 0	 Model M29 10 4 -8 -5 -4 10 2 8 -8 2 10 4 -8 4 10	 Model M30 4 0 -2 -2 0 4 -1 -1 -2 1 4 0 -2 -1 0 4
 Model M31 12 -3 -6 -2 -3 12 6 8 -6 6 12 8 -2 8 8 12	 Model M32 20 5 -10 10 5 20 -10 10 -10 -10 20 -8 10 10 -8 20	 Model M33 4 0 -3 -3 0 4 0 1 -3 0 4 1 3 -1 1 4	 Model M34 4 1 -3 -2 -1 4 1 2 3 1 4 2 -2 2 2 4	 Model M35 4 0 -2 -3 0 4 3 -2 -3 3 4 0 -3 -2 0 4	 Model M36 4 1 -2 2 1 4 -2 2 -2 2 -4 1 2 2 -1 4
 Model M37 10 0 6 -3 0 10 4 -8 6 4 10 -5 -3 -8 -5 10	 Model M38 4 0 -2 0 0 4 -3 3 -2 -3 -4 -3 0 3 -3 4	 Model M39 4 0 1 -2 0 4 -3 0 1 0 4 -2 -2 -3 -2 4	 Model M40 4 0 -2 1 0 4 -2 1 -2 -2 4 -2 1 1 -2 4	 Model M41 12 3 -9 6 3 12 -9 6 -9 9 12 -8 6 6 -8 12	 Model M42 4 0 -2 0 0 4 -3 0 -2 -3 4 -1 0 0 1 -4
 Model M43 4 1 -2 1 1 4 -2 1 -2 -4 -2 -2 1 1 -2 4	 Model M44 4 0 2 -1 0 4 0 -3 2 0 4 -2 -1 -3 -2 4	 Model M45 4 0 -2 -3 0 4 -1 0 -2 0 4 0 -3 -1 0 4	 Model M46 6 2 -3 -4 2 6 -1 -3 -3 -1 6 2 -4 -3 2 6	 Model M47 4 0 0 0 0 4 -1 -3 -3 -1 6 2 0 -1 -1 4	 Model M48 4 0 0 0 0 4 -3 0 0 4 -2 0 0 -3 -2 4

 Model M49 4 0 0 0 0 4 1 -2 0 1 4 -2 0 -2 -2 4	 Model M50 4 0 -3 -0 0 4 0 1 -3 0 4 0 0 1 0 4	 Model M51 4 0 0 0 0 4 0 -3 0 0 4 0 0 -3 0 4	 Model M52 1 0 0 0 0 1 0 0 0 0 1 0 0 0 0 1	 Model M53 4 -3 -3 -3 -3 4 1 1 -3 1 4 1 -3 1 4 1	
 Model M54 4 0 0 0 0 4 0 0 0 0 4 0 0 0 0 4	 Model M55 4 0 0 0 0 4 0 0 0 0 4 0 0 0 0 4	 Model M56 4 0 0 0 0 4 0 0 0 0 4 0 0 0 0 4	 Model M57 4 0 0 0 0 4 0 0 0 0 4 0 0 0 0 4	 Model M58 4 0 0 0 0 4 0 0 0 0 4 0 0 0 0 4	
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 Model M65 9 0 -6 -6 0 9 3 -3 -6 3 -9 -1 -6 -3 -19	 Model M66 8 0 4 35 0 8 48 60 -8 48 80 -84 35 -80 64 80	 Model M67 -14 13 11 -7 13 14 7 2 -11 7 14 13 -7 2 13 14	 Model M68 10 0 -6 3 0 10 -8 4 -6 -8 -10 -5 3 4 -5 10	 Model M69 25 0 20 -15 0 25 10 -24 20 15 25 -24 -15 -20 -24 25	 Model M70 2 1 1 -1 1 2 1 -1 1 1 2 -1 -1 -1 2 2
 Model M71 5 0 -3 4 0 5 -4 -3 -3 -4 5 0 4 -3 0 5	 Model M72 10 0 -6 0 0 10 -8 5 -6 -8 -10 -4 0 5 -4 10	 Model M73 2 1 -1 1 1 2 1 -1 -1 1 -2 -2 1 -1 -2 2	 Model M74 2 0 1 -1 0 2 1 -1 -1 1 -2 -2 -1 -1 2 2	 Model M75 4 1 2 -1 1 4 2 -4 2 2 4 -2 -1 -4 -2 4	 Model M76 5 0 3 -3 0 5 -4 4 3 -5 5 -5 -2 -4 -5 5
 Model M77 2 0 1 0 0 2 1 -2 1 1 2 -1 0 -2 1 2	 Model M78 -1 4 -2 -2 4 -1 2 2 2 -2 4 4 -2 2 -4 4	 Model M79 5 0 4 0 0 5 3 -5 4 3 5 -3 0 -5 3 5	 Model M80 2 -1 -2 -1 -1 2 1 2 -2 1 2 1 -1 2 1 2	 Model M81 2 -2 -1 -2 -2 2 1 2 -1 1 2 1 -2 2 1 2	 Model M82 2 0 0 0 0 2 -2 1 0 -2 2 1 0 1 -2 2
 Model M83 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1	 Model M84 1 0 0 0 0 1 1 1 0 1 1 1 0 1 1 1	 Model M85 1 a b c a 1 d e b d 1 f c e f 1	 Model M86 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1	 Model M87 1 0 0 0 0 1 1 1 0 1 1 1 0 1 1 1	 Model M88 1 a b c a 1 d e b d 1 f c e f 1
 Model M86 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1	 Model M87 1 0 0 0 0 1 1 1 0 1 1 1 0 1 1 1	 Model M88 1 a b c a 1 d e b d 1 f c e f 1	 Model M89 1 0 0 0 0 1 1 1 0 1 1 1 0 1 1 1	 Model M90 1 0 0 0 0 1 1 1 0 1 1 1 0 1 1 1	 Model M91 1 0 0 0 0 1 1 1 0 1 1 1 0 1 1 1

Where: $\gamma = \sqrt{107463}$
 $a = \frac{032336}{\gamma}$
 $b = 10c = \frac{100}{\sqrt{107463}}$
 $d = 10e = \frac{3}{4}, f = \frac{1}{10}$

Petr Šimeček. Gaussian representation of independence models over four random variables.
 In COMPSTAT conference, 2006.



Šimeček's Question

Does every non-empty Gaussian CI model contain a rational point?

Model M85

Where:



1	a	b	c
a	1	d	e
b	d	1	f
c	e	f	1

$$a = \frac{3}{632836} \sqrt{1107463},$$

$$b = 10c = \frac{100}{158209} \sqrt{1107463}$$

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1	a	b	c
a	1	d	e
b	d	1	f
c	e	f	1

$$\begin{pmatrix} 357 & -21 & -343 & -147 \\ -21 & 357 & 119 & 51 \\ -343 & 119 & 357 & 153 \\ -147 & 51 & 153 & 357 \end{pmatrix}$$



Complexity bounds from real geometry

Theorem (Tarski's transfer principle)

If a polynomial system $\{f_i = 0, g_j > 0, h_k \neq 0\}$ has a solution over \mathbb{R} , then it has a solution in a finite real extension of \mathbb{Q} .

→ If $\bigwedge \mathcal{P} \Rightarrow \bigvee \mathcal{Q}$ is **false**, there exists a counterexample matrix Σ with algebraic entries.

Theorem (Positivstellensatz)

A polynomial F vanishes on the basic semialgebraic set $\{f_i = 0, g_j > 0, h_k \neq 0\}$ if and only if $0 \in \text{ideal}(f_i) + \text{cone}(g_j) + \text{monoid}^2(F, g_j, h_k)$.

→ If $\bigwedge \mathcal{P} \Rightarrow \bigvee \mathcal{Q}$ is **true**, there exists an algebraic proof for it with rational coefficients.



Universality theorems

Theorem (B. 2021)

For every finite real extension \mathbb{K}/\mathbb{Q} there exists a Gaussian CI model $\mathcal{M}_{\mathbb{K}}$ such that: for every \mathbb{L}/\mathbb{Q} , $\mathcal{M}_{\mathbb{K}}$ has an \mathbb{L} -rational point if and only if $\mathbb{K} \subseteq \mathbb{L}$.

→ The answer to Šimeček's question is **NO**.

Theorem (B. 2021)

The problem of deciding whether a CI inference formula is valid for all Gaussian distributions is polynomial-time equivalent to the existential theory of the reals.

→ Solving the inference problem can be used to check if arbitrary polynomial systems have a solution over \mathbb{R} .



A bit of the proof idea

$$\begin{aligned}
 \Sigma[ij] &= x_{ij} \rightarrow \text{impose } x_{kl} = x_{km} = x_{lm} = 0 \text{ on a correlation matrix, then:} \\
 \Sigma[ij|klm] &= x_{ij}x_{kk}x_{ll}x_{mm} + x_{im}x_{jm}x_{kl}^2 - x_{im}x_{jl}x_{kl}x_{km} - x_{il}x_{jm}x_{kl}x_{km} + x_{il}x_{jl}x_{km}^2 \\
 &\quad - x_{im}x_{jm}x_{kk}x_{ll} + x_{im}x_{jk}x_{km}x_{ll} + x_{ik}x_{jm}x_{km}x_{ll} - x_{ij}x_{km}^2x_{ll} \\
 &\quad + x_{im}x_{jl}x_{kk}x_{lm} + x_{il}x_{jm}x_{kk}x_{lm} - x_{im}x_{jk}x_{kl}x_{lm} - x_{ik}x_{jm}x_{kl}x_{lm} \\
 &\quad - x_{il}x_{jk}x_{km}x_{lm} - x_{ik}x_{jl}x_{km}x_{lm} + 2x_{ij}x_{kl}x_{km}x_{lm} + x_{ik}x_{jk}x_{lm}^2 \\
 &\quad - x_{ij}x_{kk}x_{lm}^2 - x_{il}x_{jl}x_{kk}x_{mm} + x_{il}x_{jk}x_{kl}x_{mm} + x_{ik}x_{jl}x_{kl}x_{mm} \\
 &\quad - x_{ij}x_{kl}^2x_{mm} - x_{ik}x_{jk}x_{ll}x_{mm} \\
 &= x_{ij} - \sum_{k=l,m} x_{ik}x_{jk} = x_{ij} - \left\langle \begin{pmatrix} x_{ik} \\ x_{il} \\ x_{im} \end{pmatrix}, \begin{pmatrix} x_{jk} \\ x_{jl} \\ x_{jm} \end{pmatrix} \right\rangle.
 \end{aligned}$$

The rest is 19th century projective geometry. Keyword: *von Staudt constructions*.



Approximations to the inference problem



Approximations to the inference problem

Theorem (Matúš 2005)

The following relations hold for every symmetric matrix Σ :

$$\begin{aligned}\Sigma[ij|L]^2 &= \Sigma[iL] \cdot \Sigma[jL] - \Sigma[L] \cdot \Sigma[ijL] \\ \Sigma[kL] \cdot \Sigma[ij|L] &= \Sigma[L] \cdot \Sigma[ij|kL] + \Sigma[ik|L] \cdot \Sigma[jk|L]\end{aligned}$$



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The following relations hold for every symmetric matrix Σ :

$$\begin{aligned}\Sigma[ij|L]^2 &= \Sigma[iL] \cdot \Sigma[jL] - \Sigma[L] \cdot \Sigma[ijL] && \rightarrow \text{semimatroids} \\ \Sigma[kL] \cdot \Sigma[ij|L] &= \Sigma[L] \cdot \Sigma[ij|kL] + \Sigma[ik|L] \cdot \Sigma[jk|L] && \rightarrow \text{gaussoids}\end{aligned}$$

These relations define essential geometric properties of symmetric matrices in principal and almost-principal minor coordinates. Study their combinatorics!



The Gaussian CI configuration space

$$\Sigma[kL] \cdot \Sigma[ij|L] = \Sigma[L] \cdot \Sigma[ij|kL] + \Sigma[ik|L] \cdot \Sigma[jk|L]$$

The *Gaussian CI configuration space* $\mathcal{G} \subseteq \mathbb{R}^{2^n} \times \mathbb{R}^{\binom{n}{2}2^{n-2}}$ consists of all vectors of principal and almost-principal minors of $\Sigma \in \text{PD}_n$.

$$\begin{pmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1, 1, 1, 1, 1 - a^2, 1 - b^2, 1 - c^2, \\ 1 + 2abc - a^2 - b^2 - c^2, \\ a, a - bc, b, b - ac, c, c - ab \end{pmatrix} \in \mathbb{R}^{8+6}.$$



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Very wasteful encoding of a matrix, but this creates simple and useful relations on configuration vectors. The CI structure of Σ is encoded in the *zero pattern* of $c(\Sigma) \in \mathcal{G}$.



Combinatorial compatibility

$$\Sigma[kL] \cdot \Sigma[ij|L] = \Sigma[L] \cdot \Sigma[ij|kL] + \Sigma[ik|L] \cdot \Sigma[jk|L]$$

Combinatorial compatibility means “fulfilling of relations under uncertainty”:
What if we only knew that all $\Sigma[K] \neq 0$ and whether or not $\Sigma[ij|K] = 0$ for every $(ij|K)$?



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$$(ik|L) \wedge (ij|kL) \Rightarrow (ij|L)$$

⋮



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$$(ij|L) \wedge (ik|L) \Rightarrow (ij|kL) \wedge (ik|jL)$$

This yields the definition of *gaussoids*.



CI inference via SAT solvers

Since gaussoids have a finite axiomatization, a SAT solver like CaDiCaL can deduce implications under the gaussoid axioms:

$$\begin{aligned} & (12|3) \wedge (12|34) \wedge (24|1) \wedge (34|2) \\ \Rightarrow & (12|) \wedge (12|4) \wedge (24|) \wedge (24|3) \wedge (24|13) \wedge (34|) \end{aligned}$$

These conclusions are valid for all regular Gaussian distributions.



Oriented gaussoids

$$\Sigma[kL] \cdot \Sigma[ij|L] = \Sigma[L] \cdot \Sigma[ij|kL] + \Sigma[ik|L] \cdot \Sigma[jk|L]$$

What if we only knew that all $\text{sgn } \Sigma[K] = +1$ and the value of $\text{sgn } \Sigma[ij|K]$ for every $(ij|K)$?

$$+(ij|L) \wedge -(ij|kL) \Rightarrow [+(ik|L) \wedge +(jk|L)] \vee [-(ik|L) \wedge -(jk|L)]$$

→ *Oriented* and *orientable* gaussoids.



Oriented gaussoids

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$$+(ij|L) \wedge -(ij|kL) \Rightarrow [+(ik|L) \wedge +(jk|L)] \vee [-(ik|L) \wedge -(jk|L)]$$

→ *Oriented* and *orientable* gaussoids.

$$(ij|L) \wedge (kl|L) \wedge (ik|jL) \wedge (jl|ikL) \Rightarrow (ik|L)$$

$$(ij|L) \wedge (kl|iL) \wedge (kl|jL) \wedge (ij|klL) \Rightarrow (kl|L)$$

$$(ij|L) \wedge (jl|kL) \wedge (kl|iL) \wedge (ik|jL) \Rightarrow (ik|L)$$

$$(ij|kL) \wedge (ik|lL) \wedge (il|jL) \Rightarrow (ij|L)$$

$$(ij|kL) \wedge (ik|lL) \wedge (jl|iL) \wedge (kl|jL) \Rightarrow (ij|L)$$



CI inference via SAT solvers II

Running the SAT solver CaDiCaL on the definition of oriented gaussoids confirms that on their supports

$$(12|) \wedge (13|4) \wedge (14|5) \wedge (15|23) \wedge (23|5) \wedge (24|135) \wedge (34|12) \wedge (35|1) \wedge (45|2) \\ \Rightarrow \text{everything except } (25|K) \text{ for all } K.$$

Geometrically, the model is a Markov network.

Theorem (B. 2021+)

Orientable gaussoids have no finite complete axiomatization.



Selfadhesivity

Gaussian multiinformation functions satisfy a *selfadhesivity* property just like entropy vectors of discrete random variables (Matúš 2007):

Theorem (B. 2020+)

Let Σ be a positive-definite matrix on \mathbb{R}^N and $K \subseteq N$. Let M be any set with $|M| = |N|$ and $M \cap N = K$. There exists a unique positive-definite Φ on $\mathbb{R}^{N \cup M}$ such that:

- $\Phi_N = \Sigma = \Phi_M$,
- $N \perp\!\!\!\perp M \mid K$ holds for Φ .



Structural selfadhesivity

Selfadhesivity can be formulated in CI terms and all necessary properties studied above can be **strengthened** by selfadhesivity. For example:



Structural selfadhesivity

Selfadhesivity can be formulated in CI terms and all necessary properties studied above can be **strengthened** by selfadhesivity. For example:

- \mathcal{O} is a *selfadhesive* orientable gaussoid on N if for every $K \subseteq N$ and M as above there exists an orientable gaussoid $\overline{\mathcal{O}}$ on $N \cup M$ such that
 - $\overline{\mathcal{O}}|_N = \mathcal{O} = \overline{\mathcal{O}}|_M$,
 - $(N, M|K) \in \overline{\mathcal{O}}$.

This property can be decided by 2^N calls to a program which decides orientability.

Every Gaussian CI structure is a selfadhesive orientable gaussoid, but not every orientable gaussoid is selfadhesive. Similarly for semimatroids.



The search for inference rules

Inference rules help characterize the *realizable* CI structures:

- 3-variate: 11 out of 64 by Matúš 2005.
- 4-variate: 629 out of 16 777 216 by Lněnička and Matúš 2007.
- 5-variate: *open!* (out of 1 208 925 819 614 629 174 706 176)
 - 254 826 gaussoids modulo symmetry,
 - 87 792 of which are orientable semimatroids,
 - 84 434 of which are *selfadhesive* orientable semimatroids.



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Help wanted:

- Use linear programming and information inequalities.
- Non-linear information inequalities → Ahmadieh and Vinzant 2021.
- Tropical approximations and valuated gaussoids.
- Compute algebraic realization spaces.
- Find and certify real solutions to polynomial systems.



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