The algebraic geometry of Ornstein–Uhlenbeck processes in equilibrium

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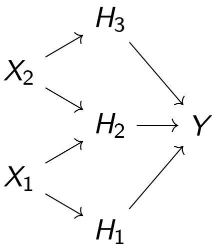
Based on joint works **arXiv:2408.00583** and **arXiv:2510.04985** with Carlos Améndola, Mathias Drton, Benjamin Hollering, Sarah Lumpp, Pratik Misra, and Daniela Schkoda.

Funded by the European Union

Causal modeling:

- We want a statistical model which captures the causal structure encoded in a directed graph $\mathcal{G} = (V, E)$.



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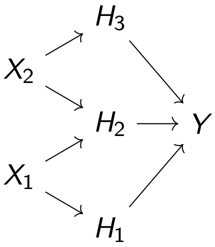

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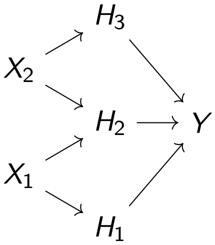
- The causal diagram may come from expert knowledge or it might be learned from observational data.



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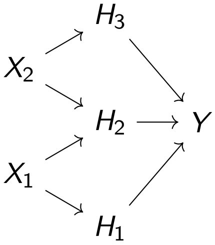

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$$X_j = \sum_{i \in \mathrm{pa}(j)} \lambda_{ij} X_i + \varepsilon_j, \quad \varepsilon_j \sim \mathcal{N}(0, \omega_j).$$

$$\begin{tikzcd}
 & H_3 \arrow[ddr] & \\
X_2 \arrow[ur] \arrow[dr] & & \\
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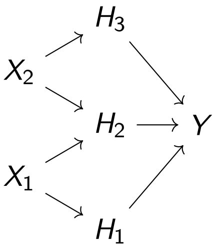

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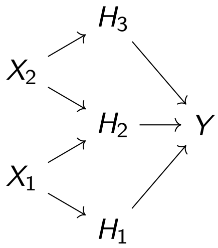

- In short: $X = \Lambda X + \varepsilon$ where Λ is $\mathcal{G}$-sparse and $\varepsilon \sim \mathcal{N}(0, \Omega)$ with Ω diagonal.
- Solutions to this system are multivariate normal distributions with zero mean and covariance matrix Σ satisfying the congruence

$$\boxed{(\mathrm{Id} - \Lambda)^{\mathsf{T}} \Sigma (\mathrm{Id} - \Lambda) = \Omega.}$$

Example

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- Since $\mathcal{G}$ is acyclic, $\det(\mathrm{Id} - \Lambda) = 1$ and $\phi_{\mathcal{G}}$ is a polynomial map!
- Using elimination theory we find that the image is cut out of PD_3 by a single quadratic: $\sigma_{13}\sigma_{22} - \sigma_{12}\sigma_{23}$.

LSEMs as statistical models

- The model $\mathcal{M}(\mathcal{G})$ is the image of the polynomial map $\phi_{\mathcal{G}} \colon \mathbb{R}^E \times \mathbb{R}^V_{>0} \to \mathrm{PD}_n$

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For many purposes we may study $\mathcal{V}(\mathcal{G})$ instead of $\mathcal{M}(\mathcal{G})$.

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- Notice that for LSEMs this is the case if and only if $\mathcal{V}(\mathcal{G}) = \mathcal{V}(\mathcal{H})$!

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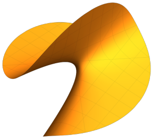

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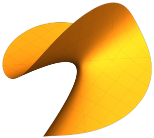

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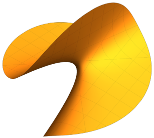

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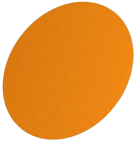

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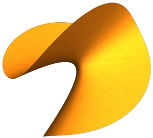

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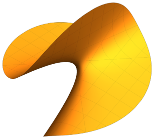

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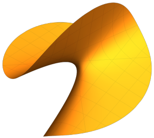

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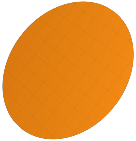

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- Same model despite **opposite** claims of causation.

Markov / model equivalence

Verma & Pearl [VP90; AMP97]

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- An edge $i \to j$ is covered if $\mathrm{pa}(j) = \mathrm{pa}(i) \cup \{i\}$.

$$\{\circ \overset{*}{\to} \circ \to \circ,\ \ \circ \overset{*}{\leftarrow} \circ \overset{*}{\to} \circ,\ \ \circ \leftarrow \circ \overset{*}{\leftarrow} \circ\} \qquad \{\circ \to \circ \leftarrow \circ\}$$

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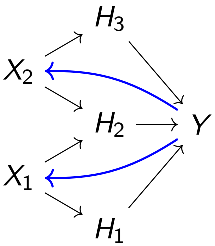

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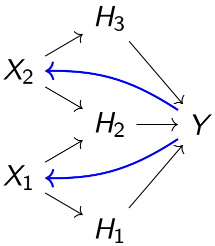

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\begin{tikzcd}
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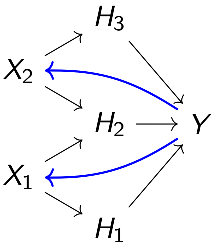
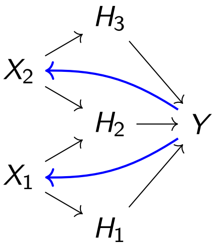
$$



Lyapunov models



- We want a statistical model which captures the causal structure of a directed graph $\mathcal{G} = (V, E)$ with cycles.
- Idea of [Fit19] and independently [VH20]: unroll the feedback loop over time:

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[dr] & X(t) \arrow[r] \arrow[dr] & X(t+1) \arrow[r] \arrow[dr] & \cdots \\
\cdots \arrow[r] \arrow[ur] & Y(t) \arrow[r] \arrow[ur] & Y(t+1) \arrow[r] \arrow[ur] & \cdots
\end{tikzcd}$$

- Formally, take the stationary distribution of the Ornstein–Uhlenbeck process

$$\mathrm{d}\mathbb{X}(t) = M\mathbb{X}(t)\mathrm{d}t + D\mathrm{d}\mathbb{W}(t), \qquad \text{(cf. } X = \Lambda X + \varepsilon\text{)}$$

where M is $\mathcal{G}$-sparse, $\mathbb{W}$ a standard Brownian motion, and D diagonal.

Parametrization of Lyapunov models

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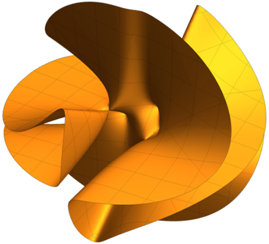
The Lyapunov model is an irreducible algebraic variety (inside PD_n)!

Beauty is in the eye of the beholder...



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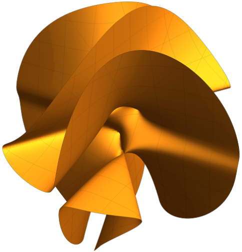

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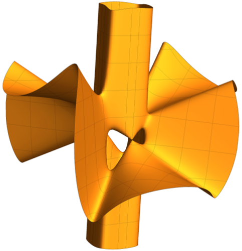

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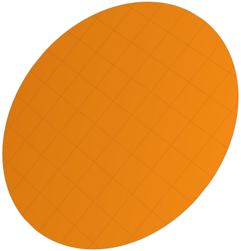

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Parameter identifiability

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- The Lyapunov equation is **also** a linear matrix equation in M, equivalent to:

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Lyapunov models are more often globally identifiable than LSEMs!

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- Analogously to LSEMs we get irreducible varieties $\mathcal{V}(\mathcal{G})$ such that:

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☞ The goal is to learn exactly what the function field $\mathbb{C}(\mathcal{V}(\mathcal{G}))$ reveals about the graph $\mathcal{G}$.

The essential ingredient: missing-edge relations

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The numerators of the $\psi_{\mathcal{G}'}(\Sigma)_{ij}$ for $i \to j \in E(\mathcal{G}') \setminus E(\mathcal{G})$ are the missing-edge relations of $\mathcal{G}$ wrt $\mathcal{G}'$. They cut out a variety $\mathcal{V}_{\mathcal{G}'}(\mathcal{G})$ such that $\mathcal{M}(\mathcal{G}) = \mathrm{PD}_n \cap \mathcal{V}_{\mathcal{G}'}(\mathcal{G})$.

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- We have $\mathcal{V}_{\mathcal{G}}^{\mathcal{G}'} \supseteq \mathcal{V}(\mathcal{G})$ and $\dim \mathcal{V}_{\mathcal{G}'}(\mathcal{G}) = \dim \mathcal{V}(\mathcal{G})$ but $\mathcal{V}_{\mathcal{G}'}(\mathcal{G})$ may be reducible.

Model equivalence via induced subgraphs

Theorem ([ABHM25])

If $\mathcal{G}$ and $\mathcal{H}$ are acyclic then $\mathcal{M}(\mathcal{G}) = \mathcal{M}(\mathcal{H})$ if and only if they have the same skeleton and $\mathcal{M}(\mathcal{G}[K]) = \mathcal{M}(\mathcal{H}[K])$ for all $K \in \binom{V}{4}$.

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Corollary

A DAG is uniquely identifiable from data (within the class of DAGs) if and only if it does not contain a super-covered edge.

A sparsest (connected) example of non-identifiability

- $\mathrm{pa}(i) \cup \{i\} = \mathrm{pa}(j)$ and $\mathrm{ch}(i) = \mathrm{ch}(j) \cup \{j\}$,
- for all $k \in \mathrm{pa}(j)$ and $l \in \mathrm{ch}(i)$ the edge $k \to l$ exists in $\mathcal{G}$,
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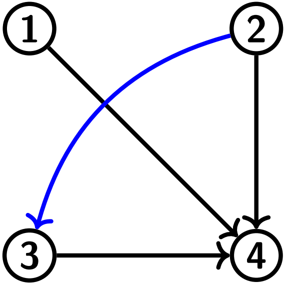

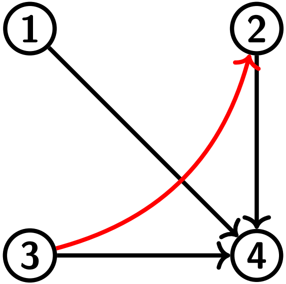

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		Lyapunov DAG models		Bayesian network models	
n	DAGs	Distinct models	Identifiable	Distinct models	Identifiable
3	25	17	13	11	4
4	543	461	423	185	59
5	29 281	27 697	26 761	8 782	2 616
6	3 781 503	3 715 745	3 665 673	1 067 825	306 117

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