

# The complexity of Gaussian conditional independence inference

Tobias Boege



Algebraic Structures in Statistical Methodology  
Mathematisches Forschungsinstitut Oberwolfach,  
4–10 December 2022

# Conditional independence models

## Definition

A *CI constraint* is a CI statement  $[\xi_i \perp\!\!\!\perp \xi_j \mid \xi_K]$  or its negation  $\neg[\xi_i \perp\!\!\!\perp \xi_j \mid \xi_K]$  constraining a random vector  $\xi$ .

# Conditional independence models

## Definition

A *CI constraint* is a CI statement  $[\xi_i \perp\!\!\!\perp \xi_j \mid \xi_K]$  or its negation  $\neg[\xi_i \perp\!\!\!\perp \xi_j \mid \xi_K]$  constraining a random vector  $\xi$ .

- ▶ How hard is it to decide if a set of constraints is consistent?
- ▶ How hard is it to **certify** consistency by exhibiting a distribution?

# Conditional independence models

## Definition

A *CI constraint* is a CI statement  $[\xi_i \perp\!\!\!\perp \xi_j \mid \xi_K]$  or its negation  $\neg[\xi_i \perp\!\!\!\perp \xi_j \mid \xi_K]$  constraining a random vector  $\xi$ .

- ▶ How hard is it to decide if a set of constraints is consistent?
- ▶ How hard is it to **certify** consistency by exhibiting a distribution?

Dual view: Conditional independence inference

## Definition

A *CI inference formula* is a Boolean formula in implication form whose variables are CI statements:  $\bigwedge \mathcal{P} \Rightarrow \bigvee \mathcal{Q}$ , where  $\mathcal{P}$  and  $\mathcal{Q}$  are sets of CI statements.

# Conditional independence models

## Definition

A *CI constraint* is a CI statement  $[\xi_i \perp\!\!\!\perp \xi_j \mid \xi_K]$  or its negation  $\neg[\xi_i \perp\!\!\!\perp \xi_j \mid \xi_K]$  constraining a random vector  $\xi$ .

















































- ▶ How hard is it to decide if a set of constraints is consistent?
- ▶ How hard is it to **certify** consistency by exhibiting a distribution?









































Dual view: Conditional independence inference

## Definition

A *CI inference formula* is a Boolean formula in implication form whose variables are CI statements:  $\bigwedge \mathcal{P} \Rightarrow \bigvee \mathcal{Q}$ , where  $\mathcal{P}$  and  $\mathcal{Q}$  are sets of CI statements.

- ▶ How hard is it to decide if an implication is valid?
- ▶ How hard is it to certify validity and with **what data**?

 Model M1 4 0 2 -2 0 4 1 -3 2 1 4 -3 -2 -3 -3 4	 Model M2 4 1 -3 2 1 4 -3 2 -3 -3 4 -3 2 -3 4	 Model M3 4 -3 -3 3 -3 4 3 2 -3 4 1 -3 2 1 4	 Model M4 4 0 1 -1 0 4 1 1 1 1 4 1 -1 1 1 4	 Model M5 4 0 2 2 0 4 2 -3 2 2 4 3 -2 -3 -3 4	 Model M6 4 0 3 -2 0 4 2 -3 3 2 4 3 -2 -3 -3 4
 Model M7 4 0 -2 2 0 4 -3 1 -2 -3 -4 1 2 -1 -1 4	 Model M8 4 1 -2 2 1 4 -3 2 -2 -3 -4 1 2 -1 -1 4	 Model M9 4 2 -3 3 2 4 -3 1 -3 -3 4 -2 3 1 -2 4	 Model M10 4 1 -2 2 1 4 -3 2 -2 -3 4 -2 2 2 -2 4	 Model M11 4 0 -3 -3 0 4 1 2 -3 1 4 2 -3 2 2 4	 Model M12 4 0 -2 1 0 4 -3 -2 -2 -3 4 0 1 -2 0 4
 Model M13 4 -1 2 -2 -1 4 -2 2 2 -2 4 -3 -2 2 -3 4	 Model M14 4 1 2 -2 1 4 -1 -2 -2 1 4 -2 -2 -2 2 4	 Model M15 6 1 4 2 1 6 5 3 -4 -5 6 3 2 3 3 6	 Model M16 4 0 2 -1 0 4 2 -3 2 2 4 3 -1 -3 3 4	 Model M17 4 0 -2 1 0 4 -3 2 -2 3 4 -2 1 2 2 4	 Model M18 4 0 -2 2 0 4 -3 -1 -2 3 4 -1 2 -1 -1 4
 Model M19 4 0 2 -2 0 4 -1 3 2 -1 4 0 -2 -3 0 4	 Model M20 4 0 -2 3 0 4 -2 1 -2 2 4 1 -3 -1 1 4	 Model M21 4 0 -2 2 0 4 -2 2 -2 2 4 1 2 -2 1 4	 Model M22 4 0 -2 2 0 4 -2 2 -2 2 4 1 2 -2 1 4	 Model M23 4 0 2 -1 0 4 1 -2 2 1 4 -2 -1 -2 2 4	 Model M24 6 0 -3 1 0 6 -4 3 -3 4 -6 3 1 3 2 6
 Model M25 8 0 4 -3 0 8 4 -7 4 4 8 -6 -3 -7 6 8	 Model M26 8 -3 -6 -6 -3 8 4 4 -6 4 8 2 -6 4 2 8	 Model M27 8 0 4 -2 0 8 2 -5 4 2 8 4 -2 -5 4 8	 Model M28 10 4 -5 -8 -4 10 2 5 -5 2 10 1 -8 5 1 10	 Model M29 10 4 -8 -5 -4 10 2 8 -8 2 10 4 -2 1 4 0	 Model M30 4 0 -2 -2 0 4 1 -1 -2 2 4 0 -2 1 0 4
 Model M31 12 -3 -6 -2 -3 12 6 8 -6 6 12 8 -2 8 8 12	 Model M32 20 5 -10 10 5 20 -10 10 -10 -10 20 -8 10 10 -8 20	 Model M33 4 0 -3 -3 0 4 0 1 -3 0 4 1 -3 1 1 4	 Model M34 4 -1 -3 -2 -1 4 1 2 -3 1 4 2 -2 2 2 4	 Model M35 4 0 -2 3 0 4 3 -2 -2 3 4 0 -3 -2 0 4	 Model M36 4 1 2 -2 1 4 2 2 -2 2 4 -1 2 2 1 4
 Model M37 10 0 6 -3 0 10 4 -8 6 4 10 -5 -3 -8 -5 10	 Model M38 4 0 -2 0 0 4 -3 3 -2 -3 4 -3 0 3 3 4	 Model M39 4 0 1 -2 0 4 0 1 1 0 4 2 -2 -3 2 4	 Model M40 4 0 -2 1 0 4 -2 1 -2 2 4 -2 1 1 2 4	 Model M41 12 3 -9 6 3 12 -9 6 -9 9 12 -8 6 6 8 12	 Model M42 4 0 -2 0 0 4 -3 0 -2 -3 4 -1 0 0 1 4
 Model M43 4 1 -2 1 1 4 -2 1 -2 4 -2 2 1 1 -2 4	 Model M44 4 0 2 -1 0 4 0 -3 2 0 4 -2 -1 -3 -2 4	 Model M45 4 0 -2 -3 0 4 0 -1 2 0 4 0 -3 -1 0 4	 Model M46 6 2 -3 4 2 6 -1 -3 -3 -1 6 2 -4 -3 2 6	 Model M47 4 0 0 0 0 4 -1 -3 0 4 -1 -3 0 3 -1 4	 Model M48 4 0 0 0 0 4 0 -3 0 4 0 -3 0 0 4 -2

 Model M49 4 0 0 0 0 4 -1 -2 0 1 4 -2 0 -2 -2 4	 Model M50 4 0 -3 0 0 4 0 1 -3 0 4 0 0 1 0 4	 Model M51 4 0 0 0 0 4 0 -3 0 0 4 0 0 -3 0 4	 Model M52 1 0 0 0 0 1 0 0 0 0 1 0 0 0 0 1	 Model M53 4 -3 -3 -3 -3 4 1 1 -3 4 0 -3 1 3 4	
 Model M54  Model M55  Model M56  Model M57  Model M58					
 Model M59 4 -3 -2 1 -3 4 -2 -2 -2 2 4 2 1 -2 2 4	 Model M60 2 0 1 -1 0 2 -1 -1 1 -1 2 -1 -1 -1 2 -1	 Model M61 5 0 4 -3 0 5 2 4 4 2 5 4 -3 4 4 5	 Model M62 8 -2 4 -7 -2 8 4 -2 -4 8 2 -7 2 2 8	 Model M63 10 0 4 -6 0 10 -3 -8 4 -3 10 0 -6 8 0 10	 Model M64 4 1 1 -2 1 4 -2 -2 -2 4 -2 -2 -2 4
 Model M65 9 0 -6 -6 0 9 3 -3 -6 3 9 -1 -6 -3 -1 9	 Model M66 32 8 44 35 8 32 48 49 -8 48 89 -84 35 89 -84 89	 Model M67 14 -15 -11 -7 -13 14 7 2 -11 7 14 13 7 2 13 14	 Model M68 10 0 -6 -3 0 10 -8 4 -6 8 10 -5 3 4 -5 10	 Model M69 25 8 20 -15 8 25 15 -20 20 15 25 -20 -15 -20 -25 25	 Model M70 1 2 1 -1 1 2 1 -1 1 1 2 -2 -1 -1 2 -2
 Model M71 5 0 -3 4 0 5 -4 3 -3 4 5 0 4 -3 0 5	 Model M72 10 6 -0 0 0 10 -8 5 -8 10 -4 0 5 -4 10	 Model M73 2 1 -1 1 1 2 1 -1 -1 1 2 -2 1 -1 2 -2	 Model M74 2 0 -1 -1 0 2 -1 -1 -1 1 2 -2 -1 1 2 -2	 Model M75 4 1 2 -1 1 4 2 4 2 2 4 -2 -1 4 -2 4	 Model M76 5 0 3 -3 0 5 4 4 2 4 2 -2 4 5 5
 Model M77 2 0 1 0 0 2 1 2 1 1 2 -1 0 -2 1 2	 Model M78 4 -1 2 -2 -1 4 -2 2 2 2 4 -4 -2 2 4 4	 Model M79 5 0 4 0 0 5 3 -5 4 3 5 -3 0 -5 3 5	 Model M80 2 -1 -2 -1 -1 2 1 2 -2 1 2 1 -1 2 1 2	 Model M81 2 -2 -1 2 -2 2 1 2 -1 1 2 1 -2 2 1 2	 Model M82 2 0 0 0 0 2 2 1 0 2 2 1 0 1 1 2
 Model M83 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1	 Model M84 1 0 0 0 0 1 1 1 0 1 1 1 0 1 1 1	 Model M85 1 a b c a 1 d e b d 1 f c e f 1	$p_{100} = \frac{3}{\sqrt{63830}} \sqrt{107843}$ $a = 10e = \frac{100}{18209} \sqrt{107843}$ $d = 10e = \frac{2}{3} \cdot f = \frac{1}{10}$		
 Model M86  Model M87  Model M88					

Petr Šimeček. "Gaussian representation of independence models over four random variables".  
In: *COMPSTAT conference*. 2006

# Gaussian conditional independence

Assume  $\xi = (\xi_i : i \in N)$  are jointly Gaussian with covariance matrix  $\Sigma \in \text{PD}_N$ .

## Definition

The polynomial  $\Sigma[K] := \det \Sigma_{K,K}$  is a *principal minor* of  $\Sigma$  and  $\Sigma[ij | K] := \det \Sigma_{iK,jK}$  is an *almost-principal minor*.

- ▶  $\Sigma$  is PD if and only if  $\Sigma[K] > 0$  for all  $K \subseteq N$ .
- ▶  $[\xi_i \perp\!\!\!\perp \xi_j \mid \xi_K]$  holds if and only if  $\Sigma[ij | K] = 0$ .
- ▶  $\mathbb{E}[\xi] = \mu$  is irrelevant.

# Very special polynomials

$$\Sigma[ij|] = x_{ij}$$

$$\Sigma[ij|k] = x_{ij}x_{kk} - x_{ik}x_{jk}$$

$$\Sigma[ij|kl] = x_{ij}x_{kk}x_{ll} - x_{il}x_{jl}x_{kk} + x_{il}x_{jk}x_{kl} + x_{ik}x_{jl}x_{kl} - x_{ij}x_{kl}^2 - x_{ik}x_{jk}x_{ll}$$

$$\begin{aligned}\Sigma[ij|klm] = & x_{ij}x_{kk}x_{ll}x_{mm} + x_{im}x_{jm}x_{kl}^2 - x_{im}x_{jl}x_{kl}x_{km} - x_{il}x_{jm}x_{kl}x_{km} + \\ & x_{il}x_{jl}x_{km}^2 - x_{im}x_{jm}x_{kk}x_{ll} + x_{im}x_{jk}x_{km}x_{ll} + x_{ik}x_{jm}x_{km}x_{ll} - \\ & x_{ij}x_{km}^2x_{ll} + x_{im}x_{jl}x_{kk}x_{lm} + x_{il}x_{jm}x_{kk}x_{lm} - x_{im}x_{jk}x_{kl}x_{lm} - \\ & x_{ik}x_{jm}x_{kl}x_{lm} - x_{il}x_{jk}x_{km}x_{lm} - x_{ik}x_{jl}x_{km}x_{lm} + 2x_{ij}x_{kl}x_{km}x_{lm} + \\ & x_{ik}x_{jk}x_{lm}^2 - x_{ij}x_{kk}x_{lm}^2 - x_{il}x_{jl}x_{kk}x_{mm} + x_{il}x_{jk}x_{kl}x_{mm} + \\ & x_{ik}x_{jl}x_{kl}x_{mm} - x_{ij}x_{kl}^2x_{mm} - x_{ik}x_{jk}x_{ll}x_{mm}\end{aligned}$$

⋮



# Gaussian CI models

## Definition

The *model* of a set of CI constraints is the set of all PD matrices which satisfy them.

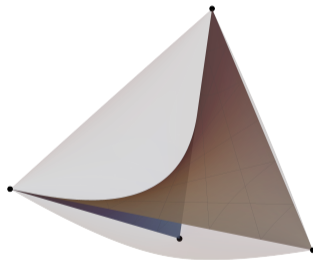
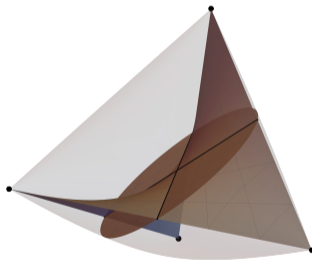


Figure: Model of  $\Sigma[12|3] = a - bc = 0$  in the  $3 \times 3$  correlation matrices.

# Gaussian CI models

## Definition

The *model* of a set of CI constraints is the set of all PD matrices which satisfy them.



**Figure:** Model of  $\Sigma[12|] = a = 0$  and  $\Sigma[12|3] = a - bc = 0$  in the  $3 \times 3$  correlation matrices.

## Models and inference

Consider two sets of CI statements  $\mathcal{P}$  and  $\mathcal{Q}$ :

$$\bigwedge \mathcal{P} \Rightarrow \bigvee \mathcal{Q}$$

## Models and inference

Consider two sets of CI statements  $\mathcal{P}$  and  $\mathcal{Q}$ :

$$\begin{array}{ccc} \bigwedge \mathcal{P} \Rightarrow \bigvee \mathcal{Q} & \iff & \mathcal{P} \cup \neg \mathcal{Q} \\ \text{is not valid} & & \text{has a point} \end{array}$$

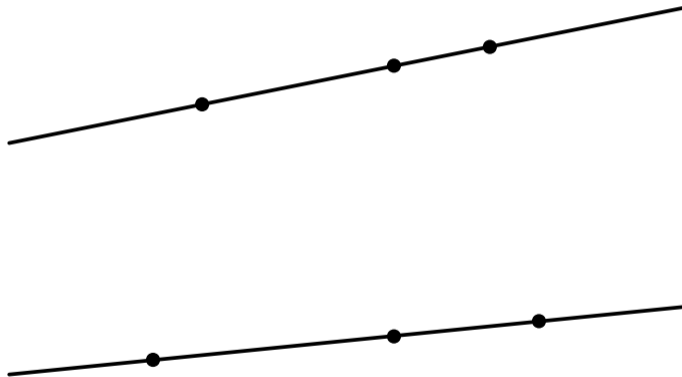
## Models and inference

Consider two sets of CI statements  $\mathcal{P}$  and  $\mathcal{Q}$ :

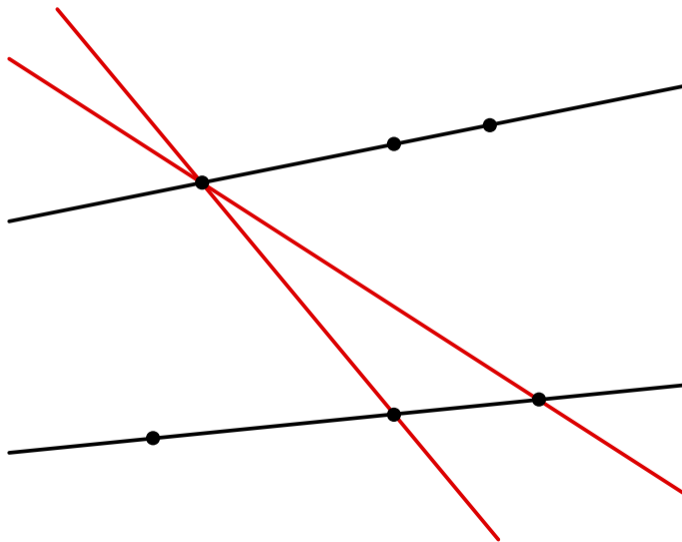
$$\begin{array}{ccc} \bigwedge \mathcal{P} \Rightarrow \bigvee \mathcal{Q} & \iff & \mathcal{P} \cup \neg \mathcal{Q} \\ \text{is not valid} & & \text{has a point} \end{array}$$

Reasoning about CI statements in normally distributed random variables is **the same** as reasoning about the vanishing of very special kinds of determinants on very special kinds of varieties inside the positive definite matrices.

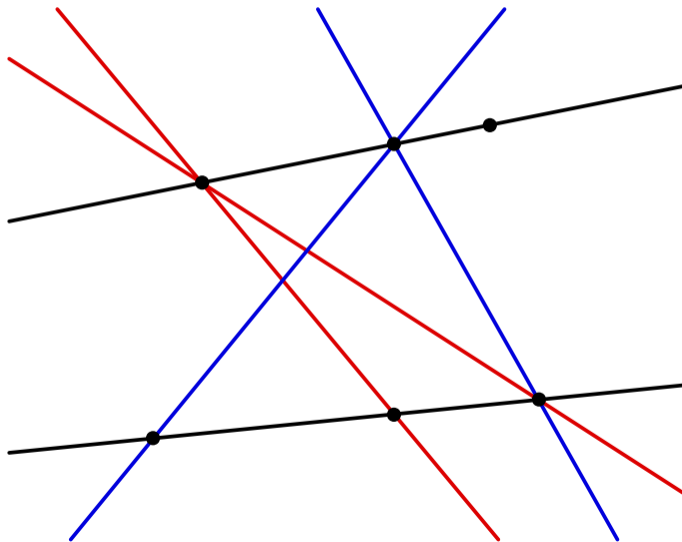
# For ancient geometers: conditional independence $\approx$ collinearity



## For ancient geometers: conditional independence $\approx$ collinearity

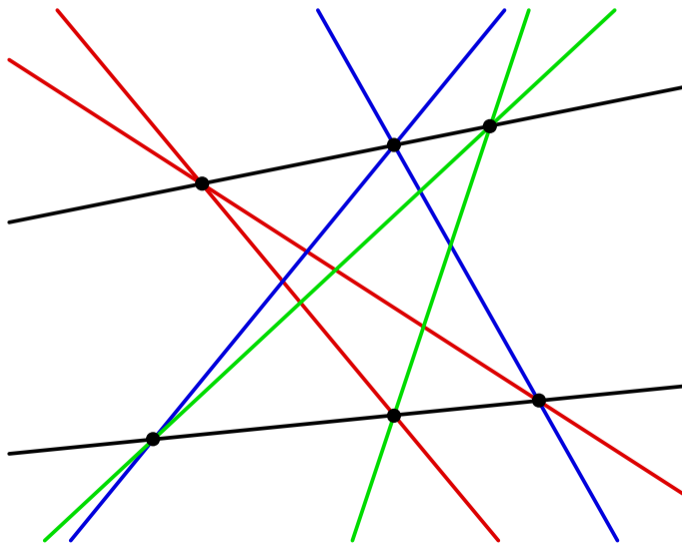


# For ancient geometers: conditional independence $\approx$ collinearity

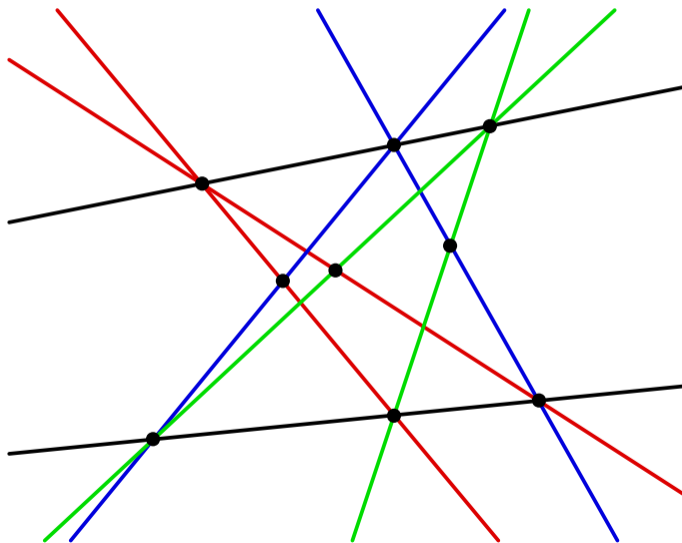




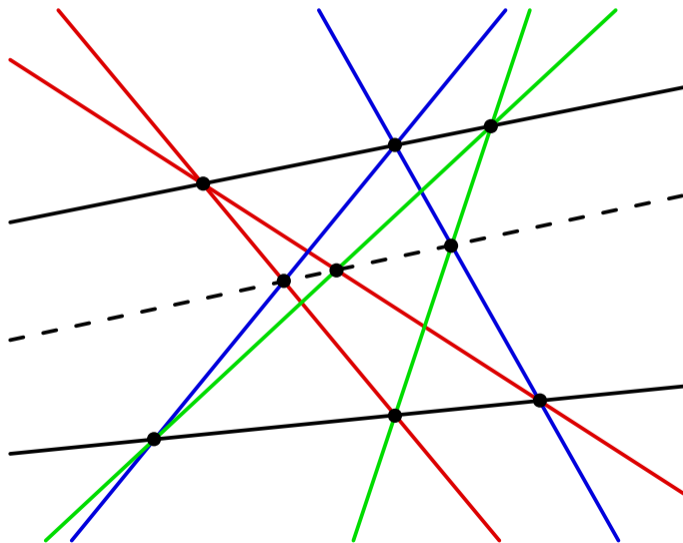
# For ancient geometers: conditional independence $\approx$ collinearity



# For ancient geometers: conditional independence $\approx$ collinearity



# For ancient geometers: conditional independence $\approx$ collinearity



## Normal form for proofs and refutations

Let  $f_i \in \mathbb{Z}[t_1, \dots, t_k]$  be integer polynomials in finitely many variables.

### Theorem (Tarski's transfer principle)

*If a polynomial system  $\{f_i \bowtie_i 0\}$ , where  $\bowtie_i \in \{=, \neq, <, \leq, \geq, >\}$ , has a solution over  $\mathbb{R}$ , then it has a solution in a finite real extension of  $\mathbb{Q}$ .*

## Normal form for proofs and refutations

Let  $f_i \in \mathbb{Z}[t_1, \dots, t_k]$  be integer polynomials in finitely many variables.

### Theorem (Tarski's transfer principle)

*If a polynomial system  $\{f_i \bowtie_i 0\}$ , where  $\bowtie_i \in \{=, \neq, <, \leq, \geq, >\}$ , has a solution over  $\mathbb{R}$ , then it has a solution in a finite real extension of  $\mathbb{Q}$ .*

→ If  $\bigwedge \mathcal{P} \Rightarrow \bigvee \mathcal{Q}$  is **false**, there exists a counterexample matrix  $\Sigma$  with algebraic entries.

$[12 | ] \wedge [12 | 3] \Rightarrow [13 | ]$  is false and a counterexample is

$$\begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1 \end{pmatrix}.$$

## Normal form for proofs and refutations

Let  $f_i, g_j, h_k \in \mathbb{Z}[t_1, \dots, t_k]$  be integer polynomials in finitely many variables.

### Theorem (Positivstellensatz)

*A polynomial system  $\{f_i = 0, g_j \geq 0, h_k \neq 0\}$  is infeasible if and only if there exist  $f \in \text{ideal}(f_i)$ ,  $g \in \text{cone}(g_j)$  and  $h \in \text{monoid}(h_k)$  such that  $g + h^2 = f$ .*

## Normal form for proofs and refutations

Let  $f_i, g_j, h_k \in \mathbb{Z}[t_1, \dots, t_k]$  be integer polynomials in finitely many variables.

### Theorem (Positivstellensatz)

*A polynomial system  $\{f_i = 0, g_j \geq 0, h_k \neq 0\}$  is infeasible if and only if there exist  $f \in \text{ideal}(f_i)$ ,  $g \in \text{cone}(g_j)$  and  $h \in \text{monoid}(h_k)$  such that  $g + h^2 = f$ .*

→ If  $\bigwedge \mathcal{P} \Rightarrow \bigvee \mathcal{Q}$  is **true**, there exists an algebraic proof for it with integer coefficients.

$[12 \mid ] \wedge [12 \mid 3] \Rightarrow [13 \mid ] \vee [23 \mid ]$  is true and a proof is the **final polynomial**

$$\Sigma[13 \mid ] \cdot \Sigma[23 \mid ] = \Sigma[3] \cdot \Sigma[12 \mid ] - \Sigma[12 \mid 3].$$

## Computer algebra proves laws of probabilistic reasoning

The following inference rule is valid for all positive definite  $5 \times 5$  matrices:

$$[12 | ] \wedge [14 | 5] \wedge [23 | 5] \wedge [35 | 1] \wedge [45 | 2] \wedge [15 | 23] \wedge [34 | 12] \wedge [24 | 135] \Rightarrow [25 | ] \vee [34 | ].$$



# Computer algebra proves laws of probabilistic reasoning

The following inference rule is valid for all positive definite  $5 \times 5$  matrices:

$$[12 | ] \wedge [14 | 5] \wedge [23 | 5] \wedge [35 | 1] \wedge [45 | 2] \wedge [15 | 23] \wedge [34 | 12] \wedge [24 | 135] \Rightarrow [25 | ] \vee [34 | ].$$

$$\begin{aligned} & [25 | ] [34 | ] \cdot [1] [2] [3] [15] = \\ & (cd^2egr + bd^2fgr - ad^2grh - 2cd^2e^2i - 2bd^2efi - 2pdfgri + 2ad^2ehi + 2pdefi^2 - 2pdqhi^2 + 2pcqi^3 + \\ & 2pdqrij - 2pbqi^2j - pcegrt + pbfgrt + pagrht + 2pce^2it - 2pcqrit + 2pbqhite - 2paehite) \cdot [12 | ] + \\ & (pdqer + pbqgr - 2pbqei) \cdot [14 | 5] - (pcdqr + p^2fgr - 2pbcqi + 2pb^2qj - 2p^2qrj) \cdot [23 | 5] + \\ & (cdqgr - 2cdqei + 2pqghi - 2pqfi^2 - pqgrj + 2pqeij - 2pe^2ft + 2pqfrit) \cdot [35 | 1] + \\ & (pd^2er - 2pbdei + p^2gri + 2pb^2et - 2p^2ert) \cdot [45 | 2] - (2pdfi - 2pbft) \cdot [15 | 23] - \\ & (d^2gr - 2d^2ei - pgrt + 2peit) \cdot [34 | 12] - 2pqi \cdot [24 | 135]. \end{aligned}$$

# Computer algebra proves laws of probabilistic reasoning

```
R = QQ[p,a,b,c,d, q,e,f,g, r,h,i, s,j, t];
X = genericSymmetricMatrix(R,p,5);
I = ideal(
  det X_{0}^{1}, det X_{0,3}^{2,3}, det X_{0,4}^{3,4},
  det X_{1,4}^{2,4}, det X_{2,0}^{4,0}, det X_{3,1}^{4,1},
  det X_{0,1,2}^{4,1,2}, det X_{2,0,1}^{3,0,1},
  det X_{1,0,2,4}^{3,0,2,4}
);
U = g*h*p*q*r*(p*t-d^2); -- [25 |][34 |] · [1][2][3][15] ∈ monoid(V)
U % I --> 0, meaning monoid(V) ∩ ideal(V) ≠ ∅ in Q[X]
-- Get a proof that U is in I:
G = gens I; -- the equations generating ideal(V)
H = U // G; -- linear combinators for U from G
U == G*H --> true
```

# Consistency checking is hard

The complexity class  $\exists\mathbb{R}$  contains all decision problems which can be reduced in polynomial time to the feasibility of a semialgebraic set:

- ▶ polynomial optimization
- ▶ computational geometry
- ▶ algebraic statistics . . .

# Consistency checking is hard

The complexity class  $\exists\mathbb{R}$  contains all decision problems which can be reduced in polynomial time to the feasibility of a semialgebraic set:

- ▶ polynomial optimization
- ▶ computational geometry
- ▶ algebraic statistics . . .

## Theorem

*The problem of deciding whether a Gaussian CI model is non-empty is  $\exists\mathbb{R}$ -complete.*

# Consistency certification is hard

Šimeček's Question (2006)

*Does every non-empty Gaussian CI model contain a rational point?*

Or: can every wrong inference rule be refuted over  $\mathbb{Q}$ ?


# Consistency certification is hard

## Šimeček's Question (2006)

*Does every non-empty Gaussian CI model contain a rational point?*

Or: can every wrong inference rule be refuted over  $\mathbb{Q}$ ?

**Model M85**



Where:

$$a = \frac{3}{632836} \sqrt{1107463},$$
$$b = 10c = \frac{100}{158209} \sqrt{1107463}$$
$$d = 10e = \frac{3}{4}, f = \frac{1}{10}$$

$$\begin{pmatrix} 1 & -1/17 & -49/51 & -7/17 \\ -1/17 & 1 & 1/3 & 1/7 \\ -49/51 & 1/3 & 1 & 3/7 \\ -7/17 & 1/7 & 3/7 & 1 \end{pmatrix}$$

# Consistency certification is hard

## Šimeček's Question (2006)

*Does every non-empty Gaussian CI model contain a rational point?*

Or: can every wrong inference rule be refuted over  $\mathbb{Q}$ ?

## Theorem

*For every finite real extension  $\mathbb{K}$  of  $\mathbb{Q}$  there exists a CI model  $\mathcal{M}$  such that  $\mathcal{M} \neq \emptyset$  but  $\mathcal{M} \cap \text{PD}_N(\mathbb{K}) = \emptyset$ .*

## Very special polynomials revisited

$\Sigma[ij|] = x_{ij} \rightarrow$  impose  $x_{kl} = x_{km} = x_{lm} = 0$  on a correlation matrix, then:

$$\begin{aligned} \Sigma[ij|klm] &= x_{ij}x_{kk}x_{ll}x_{mm} + x_{im}x_{jm}\underline{x_{kl}^2} - x_{im}x_{jl}\underline{x_{kl}x_{km}} - x_{il}x_{jm}\underline{x_{kl}x_{km}} + x_{il}x_{jl}\underline{x_{km}^2} \\ &\quad - x_{im}x_{jm}x_{kk}x_{ll} + x_{im}x_{jk}\underline{x_{km}x_{ll}} + x_{ik}x_{jm}\underline{x_{km}x_{ll}} - x_{ij}\underline{x_{km}^2}x_{ll} \\ &\quad + x_{im}x_{jl}x_{kk}\underline{x_{lm}} + x_{il}x_{jm}x_{kk}\underline{x_{lm}} - x_{im}x_{jk}\underline{x_{kl}x_{lm}} - x_{ik}x_{jm}\underline{x_{kl}x_{lm}} \\ &\quad - x_{il}x_{jk}\underline{x_{km}x_{lm}} - x_{ik}x_{jl}\underline{x_{km}x_{lm}} + 2x_{ij}\underline{x_{kl}x_{km}x_{lm}} + x_{ik}x_{jk}\underline{x_{lm}^2} \\ &\quad - x_{ij}x_{kk}\underline{x_{lm}^2} - x_{il}x_{jl}x_{kk}x_{mm} + x_{il}x_{jk}\underline{x_{kl}x_{mm}} + x_{ik}x_{jl}\underline{x_{kl}x_{mm}} \\ &\quad - x_{ij}\underline{x_{kl}^2}x_{mm} - x_{ik}x_{jk}x_{ll}x_{mm} \\ &= x_{ij} - \sum_{k=l,m} x_{ik}x_{jk} = x_{ij} - \left\langle \left( \begin{array}{c} x_{ik} \\ x_{il} \\ x_{im} \end{array} \right), \left( \begin{array}{c} x_{jk} \\ x_{jl} \\ x_{jm} \end{array} \right) \right\rangle. \end{aligned}$$

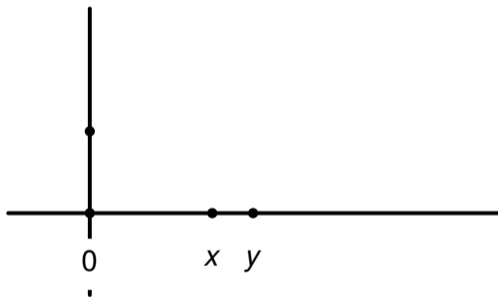


# Covariance matrix simulating a projective plane

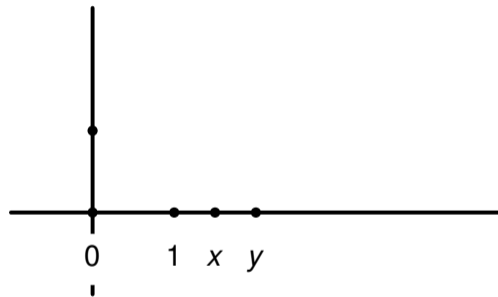
$$\begin{array}{c}
 p_1 \\
 \vdots \\
 p_n \\
 l_1 \\
 \vdots \\
 l_m \\
 x \\
 y \\
 z
 \end{array}
 \left(
 \begin{array}{ccc|ccc|ccc}
 p_1 & \dots & p_n & l_1 & \dots & l_m & x & y & z \\
 p_1^* & & \langle p, p' \rangle & & & & p_1^x & p_1^y & p_1^z \\
 & \ddots & & & \langle p, l \rangle & & & \vdots & \\
 \langle p', p \rangle & & p_n^* & & & & p_n^x & p_n^y & p_n^z \\
 \hline
 & & & l_1^* & & \langle l, l' \rangle & l_1^x & l_1^y & l_1^z \\
 & \langle l, p \rangle & & & \ddots & & & \vdots & \\
 \hline
 & & & \langle l', l \rangle & & l_m^* & l_m^x & l_m^y & l_m^z \\
 \hline
 p_1^x & & p_n^x & l_1^x & & l_m^x & x^* & 0 & 0 \\
 p_1^y & \dots & p_n^y & l_1^y & \dots & l_m^y & 0 & y^* & 0 \\
 p_1^z & & p_n^z & l_1^z & & l_m^z & 0 & 0 & z^*
 \end{array}
 \right)$$

The rest is 19<sup>th</sup> century projective geometry.

# Von Staudt constructions

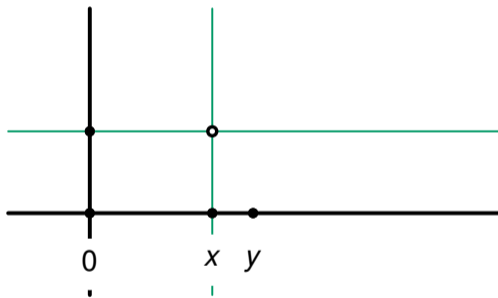


Addition

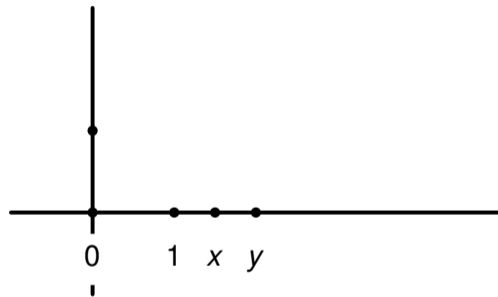


Multiplication

# Von Staudt constructions

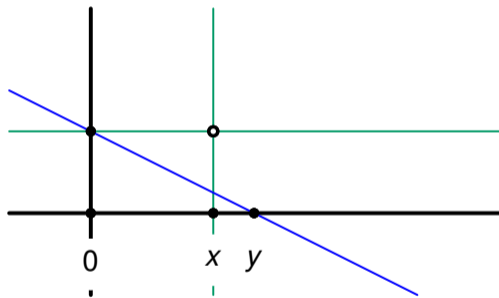


Addition

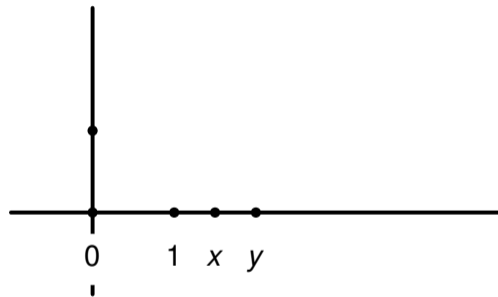


Multiplication

# Von Staudt constructions

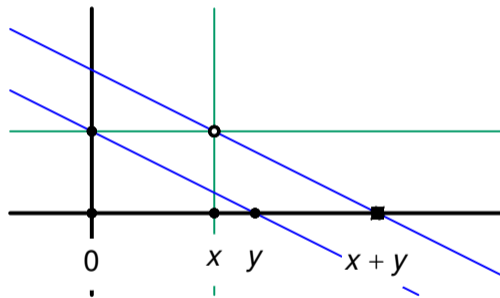


Addition

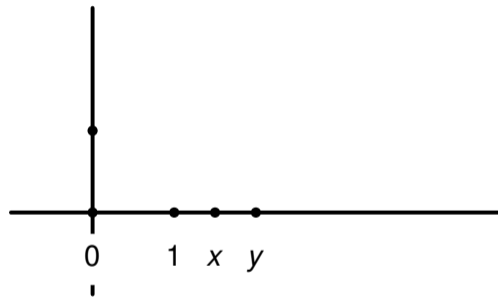


Multiplication

# Von Staudt constructions

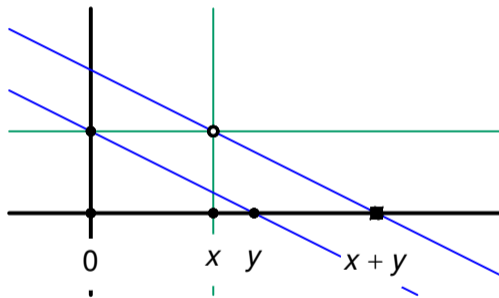


Addition

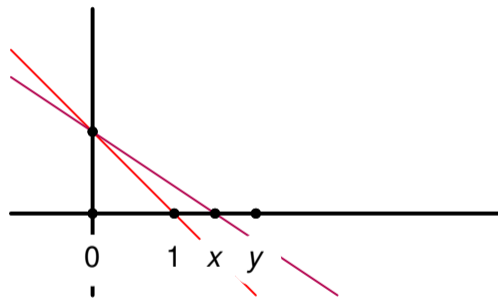


Multiplication

# Von Staudt constructions

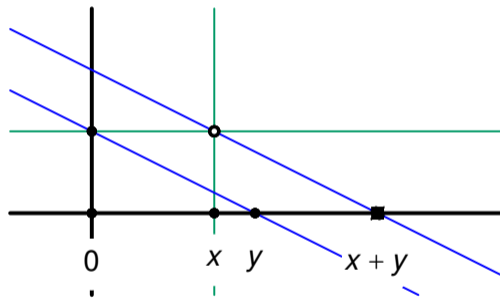


Addition

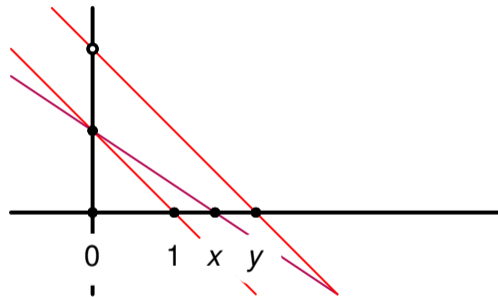


Multiplication

# Von Staudt constructions

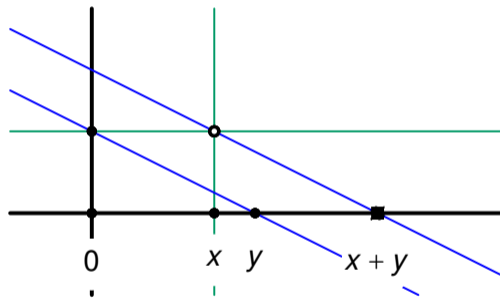


Addition

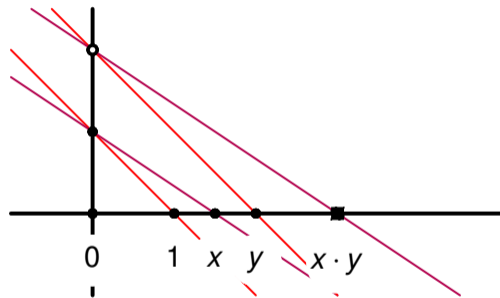


Multiplication

# Von Staudt constructions



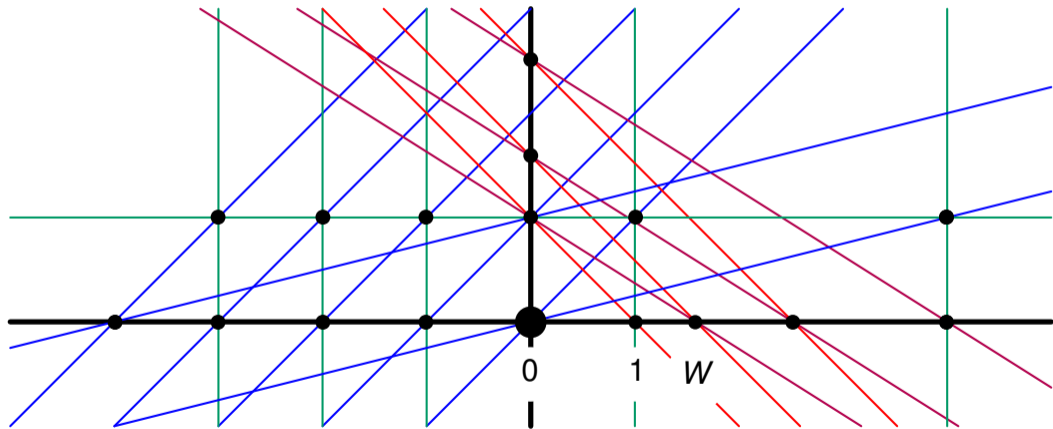
Addition



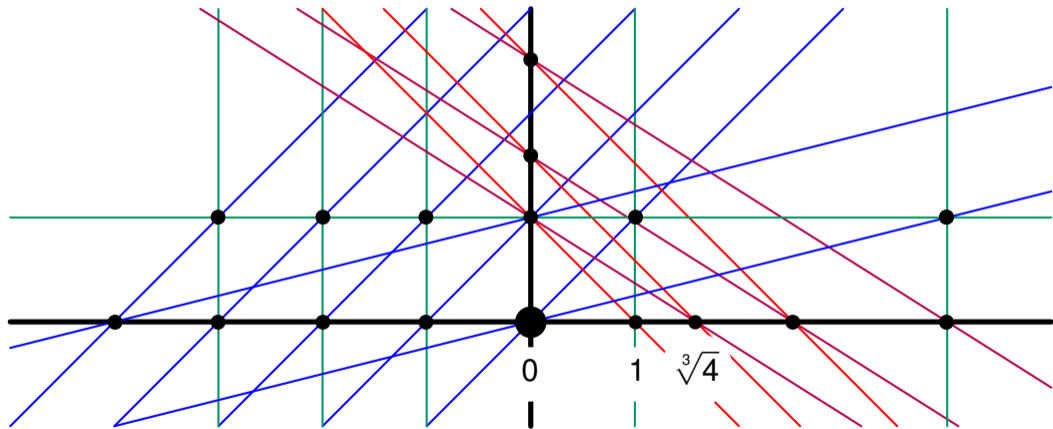
Multiplication



# Where is Waldo?



# Where is Waldo? On the cube root of 4!



## Bonus question

Matúš's Question (1999); also Sturmfels (2007)

*Can every invalid inference for **discrete** CI structures be refuted over  $\mathbb{Q}$ ?*

## Bonus question

Matúš's Question (1999); also Sturmfels (2007)

Can every invalid inference for *discrete* CI structures be refuted over  $\mathbb{Q}$ ?

```
load "simecek.m2";
```

```
f = binaryMomentMap 4; -- ... more setup
```





```
-- (Linear slice of) CI equations of binary RVs in moment coordinates
```

```
Eqns {{1},{2},{}} -->  $e_{12}$ 
```

```
Eqns {{1},{2},{3}} -->  $e_{12} - e_{13}e_{23}$ 
```

```
Eqns {{1},{2},{3,4}} -->  $e_{12} - e_{13}e_{23} - e_{14}e_{24} + e_{34}e_{1234}$  and  
-->  $e_{1234} - e_{14}e_{23} - e_{13}e_{24} + e_{12}e_{34}$ 
```

# References

-  Tobias Boege. “The Gaussian conditional independence inference problem”. PhD thesis. OvGU Magdeburg, 2022.
-  Jürgen Bokowski and Bernd Sturmfels. *Computational synthetic geometry*. Vol. 1355. Lecture Notes in Mathematics. Springer, 1989.
-  Jürgen Richter-Gebert. *Perspectives on projective geometry. A guided tour through real and complex geometry*. Springer, 2011, pp. xxii + 571.
-  Petr Šimeček. “Gaussian representation of independence models over four random variables”. In: *COMPSTAT conference*. 2006.