Polyhedra in information theory

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Entropy

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- A random vector $X \in \Delta(d_i : i \in N)$ is a random variable in $\Delta(\prod_{i \in N} d_i)$, so the definition of H extends to vectors.
- For a random vector $X = (X_i : i \in N)$ we have 2^N marginals and we collect their entropies in an entropy profile $h_X : 2^N \to \mathbb{R}$.
  - For example (X, Y) has entropy profile $(0, H(X), H(Y), H(X, Y)) \in \mathbb{R}^4$.

Entropy as information

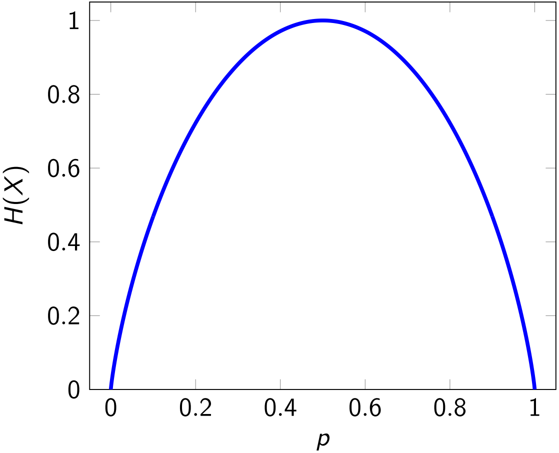

Figure: Entropy of a binary random variable X as a function of $p = \Pr[X = \text{heads}]$.

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- Graphical models in statistics and causality are defined by CI assumptions (e.g., Bayesian networks and d-separation in graphs).
- Cryptographic protocols use FD and CI constraints to specify operation and information-theoretic security (e.g., secret sharing).
- Quantities in information theory are defined by linear optimization over entropy profiles with FD and CI constraints (e.g., common information).

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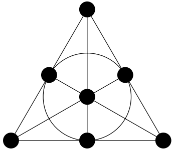

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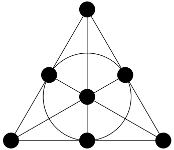

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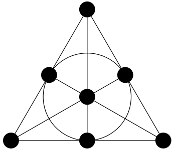

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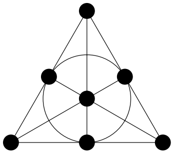

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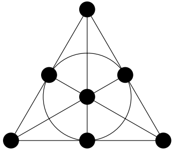

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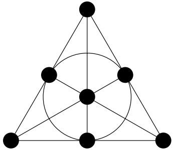

- The information ratio is $\sigma(h) = {}^{1}\!/_{h(s)} \max \{ h(p) : p \in N \}$.
- The optimal information ratio $\sigma(\mathscr{Q}) = \inf \{\sigma(h) : h \models \mathscr{Q}\}$ can be determined by linear optimization over the set of entropy profiles.

The entropy region and information inequalities

Let $\mathbf{H}_N^* \subseteq \mathbb{R}^{2^N}$ consist of all h_X where X is an N-variate discrete random vector. $\mathbf{H}_N^*$ is the image of $\bigcup_{d_1=1}^{\infty} \cdots \bigcup_{d_n=1}^{\infty} \Delta(d_1, \ldots, d_n)$ under the transcendental map $X \mapsto h_X$.

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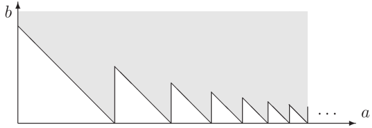
Problem

Find a description of the boundary of $\mathbf{H}_3^*$.



Shannon inequalities

- A function $h : 2^N \to \mathbb{R}$ is a polymatroid if
  - $h(\emptyset) = 0$,
  - $h(I \mid K) := h(I \cup K) - h(K) \geq 0$ ("$=$" is FD).
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Theorem ([Mat07])

$\overline{\mathbf{H}_N^*}$ *is not polyhedral for* $|N| \geq 4$.

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Which of these inequalities hold on $\overline{\mathbf{H}_4^*}$*? (Some do, some don't . . .)*

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Extension principle: Let E^M_N be the cone of an extension property and $\pi^M_N : \mathbb{R}^{2^M} \to \mathbb{R}^{2^N}$ the canonical projection. Then $\overline{\mathbf{H}^*_N} = \pi^M_N(E^M_N \cap \boxed{\overline{\mathbf{H}^*_M}})$.

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Relax: $\overline{\mathbf{H}^*_N} \subseteq \pi^M_N(E^M_N \cap \boxed{\mathbf{P}_M})$.

Extension properties: Copy lemma

- Consider $h \in \mathbf{P}_N$ and pick any $L \subseteq N$.
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$$\overline{h}|_N = h, \quad \overline{h}|_M = \sigma(h), \quad \overline{h}(N : M \mid L) = 0.$$

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- Relaxation: only require $\overline{h} \in \boxed{\mathbf{P}_{NM}}$! This gives a tighter outer bound than $\mathbf{P}_N$:

$$\mathbf{P}_N \supseteq \bigcap_{L \subseteq N} \mathbf{Copy}^L_N \supseteq \overline{\mathbf{H}^*_N}.$$

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To disprove information inequalities [KR13]:

- Take an entropy profile h.

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- Replace $\overline{\mathbf{H}_\bullet^*}$ in an extension property with $\mathbf{Q}_\bullet$.
- Project to obtain tighter polyhedral cone $\mathbf{Q}'_N \supseteq \overline{\mathbf{H}_N^*}$.
- Mix and iterate different extension properties.
- Exact polyhedral computations certify validity of new inequalities.

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Rule [43] Given:

$$
\begin{aligned}
& aI(A;B) \\
\leq\ & bI(A;B|C) + cI(A;C|B) + zI(B;C|A) \\
+\ & eI(A;B|D) + fI(A;D|B) \\
+\ & (b' + d' + z)I(B;D|A) + hI(C;D) \\
+\ & iI(C;D|A) + zI(C;D|B)
\end{aligned}
$$

and

$$
\begin{aligned}
& a'I(A;B) \\
\leq\ & b'I(A;B|C) + c'I(A;C|B) + d'I(B;C|A) \\
+\ & e'I(A;B|D) + f'I(A;D|B) + g'I(B;D|A) \\
+\ & h'I(C;D) + i'I(C;D|A) + j'I(C;D|B)
\end{aligned}
$$

Get:

$$
\begin{aligned}
& (a + a' + z)I(A;B) \\
\leq\ & (a + b + c + f + b' + 2z)I(A;B|C) \\
+\ & (-a + b + c + e + c' + z)I(A;C|B) \\
+\ & (d' + z)I(B;C|A) + (e + e' + z)I(A;B|D) \\
+\ & (f + f')I(A;D|B) \\
+\ & (-a' + b' + e' + g' + i')I(B;D|A) \\
+\ & (h + h' + z)I(C;D) + (i + i')I(C;D|A) \\
+\ & (j')I(C;D|B)
\end{aligned}
$$

Using: RS is copy of CD over AB
Substitutions: A C R S; AD B R S

Abbreviated Proof of (75): T: D-copy of A over BCRS.
L1: -a.c. +c.d. +r.cd.a +c.s.a +b.d.s +a.bs.d +2a.cr.bs +a.bs.cr +d.r.abcs +d.s.abcr
SL1: d.t.a +c.d.t +a.t.cd +c.r.t +a.t.cr +d.r.act +b.t.acdr +a.t.bs +c.s.at +b.t.acs +d.t.s +a.s.dt +b.d.ast +c.t.abds +a.r.bcst +r.ad.bcst +s.ad.bcrt +d.t.abcrs C2L1: 3t.ad.bcrs
S: C-copy of A over BDR.
L2: -2a.c. +2c.d. +a.b.cr +2a.c.br +c.ar.b +a.b.dr +4a.d.br +2a.br.d +2d.br.a +2r.cd.a +d.r.abc
SL2: c.s.b +a.b.cs +c.d.s +a.s.cd +d.s.abc +3a.s.br +3c.s.br +c.r.abs +d.r.s +a.s.dr +d.r.abs +d.br.as +c.r.ads +b.s.acdr +2c.s.abdr +2d.s.abcr
C2L2: 7s.ac.bdr
R: D-copy of C over AB.
S: c.r.a +3c.r.b +d.r.a +7d.r.b +c.d.r +2b.r.acd +r.ab.cd +9c.r.abd +3d.r.abc
C2: 16r.cd.ab

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