Information inequalities and polynomial optimization

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- What is the infimum over PD_4 of

$$[\varphi] = \frac{[13][14][23][24][34]}{[12][3][4][134][234]}$$
$$= \frac{(pr - b^2)(ps - c^2)(qr - d^2)(qs - e^2)(rs - f^2)}{rs(pq - a^2)(prs - sb^2 - rc^2 + 2bcf - pf^2)(qrs - sd^2 - re^2 + 2def - qf^2)} \,?$$

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Alternatively: what is the boundary of the convex cone $\mathcal{A}_n$ like?

Some elementary inequalities

- For $I, J, K \subseteq N$ the Koteljanskii inequality says

$$[I : J \mid K] := \frac{\Sigma[I \cup K] \cdot \Sigma[J \cup K]}{\Sigma[I \cup J \cup K] \cdot \Sigma[K]} \geq 1.$$

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- The φ functional encodes a trade-off between conditional independencies:

$$[\varphi] = \frac{[1 : 2][3 : 4 \mid 1][3 : 4 \mid 2]}{[3 : 4]}.$$

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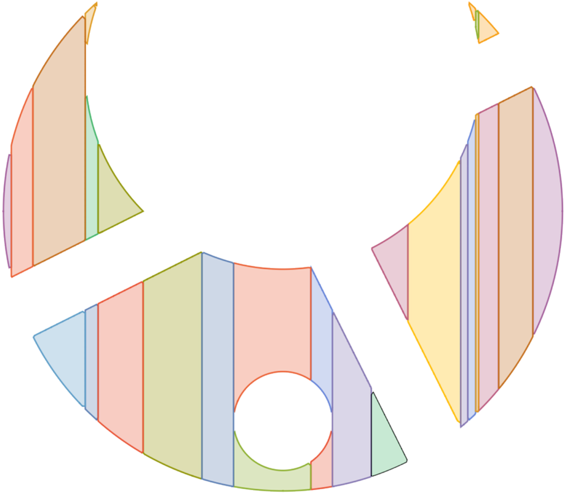
- $\alpha \in \mathcal{A}_n \iff \langle \alpha, h \rangle \geq 0 \; \forall h \in \gamma_n^* \iff \Sigma[\alpha] \geq 1 \; \forall \Sigma \in \mathrm{PD}_n$.
- If α is $\mathbb{Q}$-valued, this is decidable.

Cylindrical algebraic decomposition



https://mathematica.stackexchange.com/a/83166

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- Information inequalities for discrete RVs subsume rank inequalities for algebraic matroids, inequalities on subgroup indices of finite groups, and more.
- Can use PD matrices to disprove such inequalities.

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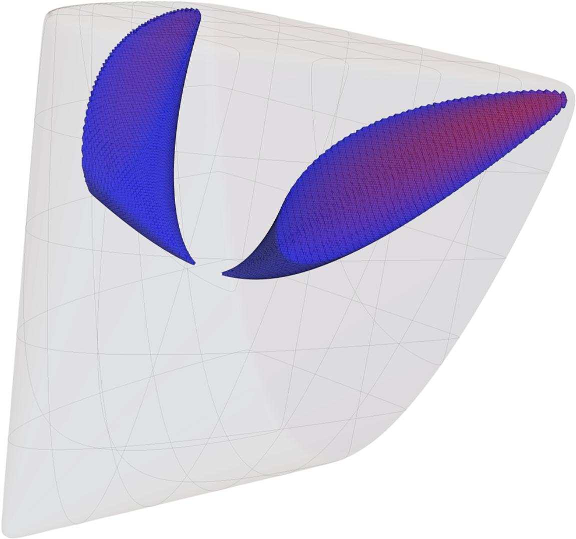
- The curve

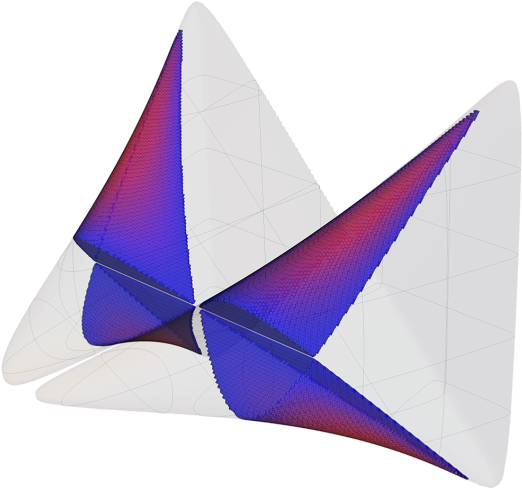
$$\Sigma_t = \begin{pmatrix} 1 & t^3 & t & t \\ t^3 & 1 & t & t \\ t & t & 1 & t \\ t & t & t & 1 \end{pmatrix}$$

is positive definite for $t \to 1^-$ and has $\Sigma_t[\varphi] \to {}^{16}/_{27}$.

The Wings of Ingleton



Conditional inequalities



The infimum of $[\varphi]$ subject to $[3:4 \mid 1] = [3:4 \mid 2] = 1$ is $^{27}/_{32}$.

The GMM functionals

$$[\varphi_k] := [\varphi] \cdot \left(\frac{[13][34]}{[3][134]} \cdot \frac{[14][34]}{[4][134]} \cdot \frac{[23][34]}{[3][234]} \cdot \frac{[24][34]}{[4][234]} \right)^k, \text{ for real } k \geq 0$$

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The infimum of $[\varphi_k]$ *over* PD_4 *is given by*

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- Have a Lean proof thanks to Thomas Kahle and Claude Code.

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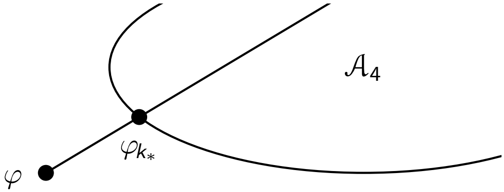

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- Good (tropical) heuristics for disproving inequalities?
- Forthcoming work on stable conditional independence implication:

"If $A \implies B$ but A holds only approximately,
does B hold up to an error of the same magnitude?"

References I

[ARVY25] Abeer Al Ahmadieh, Felipe Rincón, Cynthia Vinzant, and Josephine Yu. *Tropicalizing Principal Minors of Positive Definite Matrices*. 2025. arXiv: 2410.11220 [math.CO].

[Cha03] Terence H. Chan. "Balanced information inequalities". In: *IEEE Trans. Inf. Theory* 49.12 (2003), pp. 3261–3267. DOI: 10.1109/TIT.2003.820037.

[CGY12] Terence H. Chan, Dongning Guo, and Raymond W. Yeung. "Entropy functions and determinant inequalities". In: *2012 IEEE International Symposium on Information Theory Proceedings*. Institute of Electrical and Electronics Engineers (IEEE), 2012, pp. 1251–1255. DOI: 10.1109/ISIT.2012.6283057.

[GMM17] Arley Gómez, Carolina Mejía, and Juan Andrés Montoya. "Defining the almost-entropic regions by algebraic inequalities". In: *Int. J. Inf. Coding Theory* 4.1 (2017), pp. 1–18. DOI: 10.1504/IJICOT.2017.081456.

[HJ08] H. Tracy Hall and Charles R. Johnson. *Bounded Ratios of Products of Principal Minors of Positive Definite Matrices*. 2008. arXiv: 0806.2645 [math.CO].

References II

[JB93] Charles R. Johnson and Wayne W. Barrett. “Determinantal inequalities for positive definite matrices”. In: *Discrete Math.* 119.1-3 (1993), pp. 97–106. DOI: 10.1016/0012-365X(93)90119-E.

[Stu21] Milan Studený. “Conditional independence structures over four discrete random variables revisited: conditional ingleton inequalities”. In: *IEEE Trans. Inf. Theory* 67.11 (2021), pp. 7030–7049. ISSN: 0018-9448. DOI: 10.1109/TIT.2021.3104250.