Colored Gaussian DAG models

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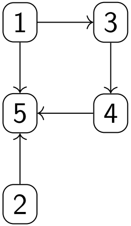
Gaussian DAG models

- A linear structural equation model defines random variables X recursively via a directed acyclic graph $G = (V, E)$ and Gaussian noise:

$$X_j = \sum_{i \in \mathrm{pa}(j)} \lambda_{ij} X_i + \varepsilon_j, \quad \varepsilon_j \sim \mathcal{N}(0, \omega_j).$$



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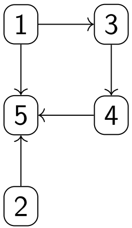

- The vector X is again Gaussian with mean zero. Since G is acyclic, we can solve for the covariance matrix Σ:

$$\Sigma = (I - \Lambda)^{-T} \Omega (I - \Lambda)^{-1}, \quad \text{with } \Lambda \in \mathbb{R}^E \text{ and } \Omega = \mathrm{diag}(\omega).$$

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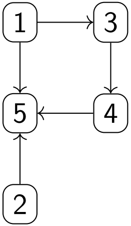

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- All such matrices form the model $\mathcal{M}(G)$.

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- Model equivalence $\mathcal{M}(G) = \mathcal{M}(H)$ is combinatorially characterized: if and only if G and H have the same skeleton and v-structures.

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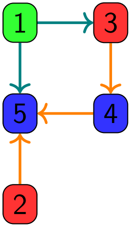
- In a colored Gaussian DAG model, the vertices and edges of G are partitioned into color classes via a coloring function $c : V \sqcup E \to C$.



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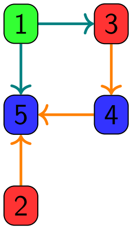

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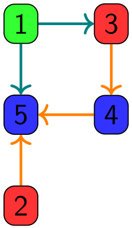

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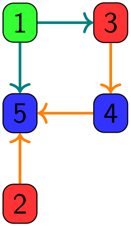

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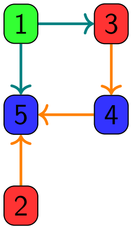
- Coloring reduces Markov-equivalence classes which eases causal discovery.



Parameter identifiability revisited

- It follows from the recursive factorization and some linear algebra that

$$\omega_j = \frac{|\Sigma_{j \cup \mathrm{pa}(j)}|}{|\Sigma_{\mathrm{pa}(j)}|}, \quad \lambda_{ij} = \frac{|\Sigma_{ij|\mathrm{pa}(j)\setminus i}|}{|\Sigma_{\mathrm{pa}(j)}|}.$$

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- A set A is identifying for a vertex j resp. edge ij if

$$\omega_j = \omega_{j|A}(\Sigma) \text{ resp. } \lambda_{ij} = \lambda_{ij|A}(\Sigma)$$

for all $\Sigma \in \mathcal{M}(G)$.

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Let $G = (V, E)$ be a DAG. Then:

- $\omega_j = \omega_{j|A}(\Sigma)$ *for every* $\Sigma \in \mathcal{M}(G)$ *if and only if* $\mathrm{pa}(j) \subseteq A \subseteq V \setminus \overline{\mathrm{de}}(j)$. *[WD23]*

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- The polynomials $\mathrm{vcr}(i|A, j|B) = |\Sigma_A||\Sigma_B|(\omega_{i|A} - \omega_{j|B})$ resp. $\mathrm{ecr}(ij|A, kl|B) = |\Sigma_A||\Sigma_B|(\lambda_{ij|A} - \lambda_{kl|B})$ vanish on the model $\mathcal{M}(G, c)$ whenever $c(i) = c(j)$ resp. $c(ij) = c(kl)$ and A and B are identifying.

Model geometry

Theorem

For every colored DAG (G, c) *the model* $\mathcal{M}(G, c)$ *is an irreducible variety and a smooth submanifold of* PD_V. *It is diffeomorphic to an open ball of dimension* $\mathrm{vc} + \mathrm{ec}$ *(the number of vertex- and edge-color classes).*

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The vanishing ideal $P_{G,c}$ of $\mathcal{M}(G, c)$ is $(I_G + I_c) : S_G$ where:

- $I_G = \langle |\Sigma_{ij|\mathrm{pa}(j)}| : ij \notin E \rangle$ *is the conditional independence ideal of* G,
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- Resolves the colored generalization of a conjecture of Sullivant; see also [RP14].

Implicitization up to saturation

Lemma

Let R, R' be rings, $S \subseteq R$ multiplicatively closed, and:

- *maps $\phi : R \to R'$ and $\psi : R' \to S^{-1}R$ with $\psi \circ \phi = \mathrm{id}_R$,*
- *for a prime ideal $I' = \langle f_1, \dots, f_k \rangle$, write $\psi(f_i) = g_i/h_i$ and set $J = \langle g_i \rangle$.*

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- The lemma computes the vanishing ideal up to a saturation of rationally identifiable models with additional equation constraints $\to$ colored undirected graphical models. What else?
- Knowing a parametrization and generators for the vanishing ideal up to saturation is sufficient in practice for model distinguishability.
- Conceivable to extend to inequalities.

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Fix a colored DAG (G, c) and $\Sigma \in \mathcal{M}(G, c)$.

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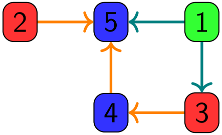
- The example on the right colors vertices and edges.
  The generic matrix in the model satisfies $1 \perp\!\!\!\perp 4 \mid 5$.
  No faithful distribution!



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Theorem

If (G, c) and (H, d) are edge-colored DAGs, then $\mathcal{M}(G, c) = \mathcal{M}(H, d)$ implies that (G, c) and (H, d) are similar. In particular, if every edge is in a color class of size at least 2, edge directions are uniquely determined.

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