

Tobias Boege

Two universality results for Gaussian CI models

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Institut für Algebra und Geometrie
Otto-von-Guericke-Universität Magdeburg



DFG-Graduiertenkolleg
MATHEMATISCHE
KOMPLEXITÄTSREDUKTION

Gaussian conditional independence

Consider random variables $(\xi_i)_{i \in N} \sim \mathcal{N}(\mu, \Sigma)$. The *conditional independence (CI) statement* $\xi_i \perp\!\!\!\perp \xi_j \mid \xi_K$ conveys, informally, that if ξ_K is known, then learning the value of one variable does not give any information about the other one.

Definition

The polynomial $\Sigma[K] := \det \Sigma_{K,K}$ is a *principal minor* of Σ and $\Sigma[ij|K] := \det \Sigma_{iK,jK}$ is an *almost-principal minor*.

If Σ is positive-definite, then $\Sigma[K] > 0$, and $\xi_i \perp\!\!\!\perp \xi_j \mid \xi_K$ holds if and only if $\Sigma[ij|K] = 0$.



Almost-principal minors

$$\Sigma[ij] = x_{ij}$$

$$\Sigma[ij|k] = x_{ij}x_{kk} - x_{ik}x_{jk}$$

$$\Sigma[ij|kl] = x_{ij}x_{kk}x_{ll} - x_{il}x_{jl}x_{kk} + x_{il}x_{jk}x_{kl} + x_{ik}x_{jl}x_{kl} - x_{ij}x_{kl}^2 - x_{ik}x_{jk}x_{ll}$$

$$\begin{aligned}\Sigma[ij|klm] = & x_{ij}x_{kk}x_{ll}x_{mm} + x_{im}x_{jm}x_{kl}^2 - x_{im}x_{jl}x_{kl}x_{km} - x_{il}x_{jm}x_{kl}x_{km} + x_{il}x_{jl}x_{km}^2 \\ & - x_{im}x_{jm}x_{kk}x_{ll} + x_{im}x_{jk}x_{km}x_{ll} + x_{ik}x_{jm}x_{km}x_{ll} - x_{ij}x_{km}^2x_{ll} \\ & + x_{im}x_{jl}x_{kk}x_{lm} + x_{il}x_{jm}x_{kk}x_{lm} - x_{im}x_{jk}x_{kl}x_{lm} - x_{ik}x_{jm}x_{kl}x_{lm} \\ & - x_{il}x_{jk}x_{km}x_{lm} - x_{ik}x_{jl}x_{km}x_{lm} + 2x_{ij}x_{kl}x_{km}x_{lm} + x_{ik}x_{jk}x_{lm}^2 \\ & - x_{ij}x_{kk}x_{lm}^2 - x_{il}x_{jl}x_{kk}x_{mm} + x_{il}x_{jk}x_{kl}x_{mm} + x_{ik}x_{jl}x_{kl}x_{mm} \\ & - x_{ij}x_{kl}^2x_{mm} - x_{ik}x_{jk}x_{ll}x_{mm}\end{aligned}$$

⋮



Models and inference

Definition

A *CI constraint* is a CI statement $\xi_i \perp\!\!\!\perp \xi_j \mid \xi_K$ or its negation $\neg(\xi_i \perp\!\!\!\perp \xi_j \mid \xi_K)$. They are *algebraic conditions* on the entries of Σ , equivalent to vanishing or non-vanishing of the almost-principal minors $\Sigma[ij|K]$.

Definition

The *model* of a set of CI constraints is the set of all positive-definite matrices which satisfy the constraints.



Models and inference

Consider two sets of CI statements \mathcal{L} and \mathcal{M} :

$$\begin{array}{ccc} \bigwedge \mathcal{L} \Rightarrow \bigvee \mathcal{M} & \iff & \mathcal{L} \cup \neg \mathcal{M} \\ \text{is not valid} & & \text{has a model} \end{array}$$

Reasoning about relevance statements in normally distributed random variables is **the same** as reasoning about the vanishing of very special kinds of determinants in the positive-definite matrices.



Model M1 4 0 2 -2 0 4 1 -3 2 1 4 -3 -2 -3 -4	Model M2 4 1 -3 -2 1 4 -3 2 -3 4 3 2 -2 2 -4	Model M3 4 -3 -3 -3 -3 4 3 2 -3 3 4 1 -3 2 1 4	Model M4 4 0 1 -1 0 4 1 1 1 1 4 1 -1 1 1 4	Model M5 4 0 2 -2 0 4 2 -3 2 2 4 -3 -2 -3 -4	Model M6 4 0 3 -2 0 4 -2 -3 3 2 4 -3 -2 -3 -4
Model M7 4 0 -2 -2 0 4 -3 1 -2 -3 4 -1 2 1 -1 4	Model M8 4 1 -2 2 1 4 -3 2 -2 -3 4 -1 2 2 -1 4	Model M9 4 2 -3 3 2 4 -3 1 -3 -3 4 -2 3 1 -2 4	Model M10 4 1 -2 2 1 4 -3 2 -2 -3 4 -1 2 2 -2 4	Model M11 4 0 -3 -3 0 4 1 2 -3 1 4 2 -3 2 2 4	Model M12 4 0 -2 -1 0 4 -3 -2 -2 -3 4 0 1 -2 0 4
Model M13 4 -1 2 -2 -1 4 -2 2 -2 4 -3 2 -2 2 -3 4	Model M14 4 1 2 -2 1 4 -1 -2 -2 1 4 2 -2 -2 -4	Model M15 6 1 4 2 1 6 -5 3 -4 -5 6 -3 2 3 -3 6	Model M16 4 0 2 -1 0 4 2 -3 2 2 4 -3 -1 -3 -3 4	Model M17 4 0 -2 1 0 4 -3 2 -2 -3 4 -2 1 2 -2 4	Model M18 4 0 -2 2 0 4 -3 -1 -2 4 -3 -1 2 -1 -1 4
Model M19 4 0 2 -2 0 4 -1 -3 2 -1 4 0 -2 -3 0 4	Model M20 4 0 -2 -3 0 4 2 -1 -2 2 4 -1 -3 -1 1 4	Model M21 4 0 -2 2 0 4 -2 2 -2 2 4 -1 2 2 -1 4	Model M22 4 0 -2 2 0 4 -2 2 -2 2 4 -1 2 2 -1 4	Model M23 4 0 2 -1 0 4 -1 2 2 1 4 2 -1 -2 2 4	Model M24 6 0 -3 1 0 6 -4 3 -3 4 -6 2 1 3 -2 6
Model M25 8 0 4 -3 0 8 4 -7 4 4 8 -6 -3 -7 -6 8	Model M26 8 -3 -6 -6 -3 8 4 4 -6 4 8 2 -6 4 2 8	Model M27 8 0 4 -2 0 8 2 -5 4 2 8 4 -2 -5 -8 8	Model M28 10 -4 -5 -8 -4 10 2 5 -5 2 10 1 -8 5 1 10	Model M29 10 -4 -8 -5 -4 10 2 8 -8 2 10 4 -5 8 4 10	Model M30 4 0 -2 -2 0 4 -1 -1 -2 1 4 0 -2 -1 0 4
Model M31 12 -3 -6 -2 -3 12 6 8 -6 6 12 8 -2 8 8 12	Model M32 20 5 -10 10 5 20 -10 10 -10 -10 20 -8 10 10 -8 20	Model M33 4 0 -3 -3 0 4 0 1 -3 0 4 1 -3 1 1 4	Model M34 4 -1 -3 -2 -1 4 1 2 -3 1 4 2 -2 2 2 4	Model M35 4 0 -2 -3 0 4 3 2 -3 3 4 0 -3 2 0 4	Model M36 4 1 -2 2 1 4 -2 2 -2 2 4 -1 2 2 1 4
Model M37 10 0 6 -3 0 10 4 -8 6 4 10 -5 -3 -8 -5 10	Model M38 4 0 -2 0 0 4 -3 3 -2 -3 -4 -3 0 3 -3 4	Model M39 4 0 1 -2 0 4 0 -3 1 0 4 -2 -2 -3 -2 4	Model M40 4 0 -2 1 0 4 -2 1 -2 -2 -4 -2 1 1 -2 4	Model M41 12 3 -9 6 3 12 -9 6 -9 -9 12 -8 6 6 -8 12	Model M42 4 0 -2 0 0 4 -3 0 -2 -3 -4 -1 0 0 1 -4
Model M43 4 1 -2 1 1 4 -2 1 -2 -4 -2 -2 1 1 -2 4	Model M44 4 0 2 -1 0 4 0 -3 2 0 4 -2 -1 -3 -2 4	Model M45 4 0 -2 -3 0 4 0 -1 -2 0 4 0 -3 -1 0 4	Model M46 6 2 -3 -4 2 6 -1 -3 -3 -1 6 2 -4 -3 2 6	Model M47 4 0 0 0 0 4 -1 -3 -3 -1 6 2 0 -1 -1 4	Model M48 4 0 0 0 0 4 0 -3 0 0 4 -2 0 -3 -2 4

Model M49 4 0 0 0 0 4 1 -2 0 1 4 -2 -2 -2 4	Model M50 4 0 -3 0 0 4 0 1 -3 0 4 0 0 1 0 4	Model M51 4 0 0 0 0 4 0 -3 0 0 4 0 0 -3 0 4	Model M52 1 0 0 0 0 1 0 0 0 0 1 0 0 0 0 1	Model M53 4 -3 -3 -3 -3 4 1 1 -3 1 4 1 -3 1 1 4	
Model M54 4 0 -2 -2 0 4 -3 1 -2 -3 4 -1 2 1 -1 4	Model M55 4 1 -2 2 1 4 -3 2 -2 -3 4 -1 2 2 -1 4	Model M56 4 2 -3 3 2 4 -3 1 -3 -3 4 -2 3 1 -2 4	Model M57 4 0 -3 -3 0 4 1 2 -3 1 4 2 -3 2 2 4	Model M58 4 0 -2 1 0 4 -3 -2 -2 -3 4 0 1 -2 0 4	
Model M59 4 -3 -2 1 -3 4 2 -2 -2 2 4 2 1 -2 2 4	Model M60 2 0 1 -1 0 2 -1 -1 -1 1 -2 1 -1 -1 -2	Model M61 5 0 4 -3 0 5 2 -4 4 2 5 -4 -3 -4 4 5	Model M62 8 -2 -4 -7 -2 8 4 -2 -4 8 2 -2 -7 -2 2 8	Model M63 10 0 4 -6 0 10 -3 -8 -3 10 0 -8 -6 -8 0 10	Model M64 4 1 1 -2 1 4 -2 -2 -2 4 -2 -2 -2 -2 4
Model M65 9 0 -6 -6 0 9 3 -3 -6 3 -9 -1 -6 -3 -1 9	Model M66 8 0 4 -6 0 8 4 -6 -6 4 8 -6 -6 -6 8 8	Model M67 -13 14 7 -7 14 -13 14 7 -7 14 -13 14 7 -7 14 13	Model M68 10 0 -6 -3 0 10 -8 4 -6 -8 -10 -5 3 4 -5 10	Model M69 25 0 20 -15 0 25 10 -20 20 15 25 -24 -15 -20 -24 25	Model M70 2 1 1 -1 1 2 1 -1 -1 1 2 -1 -1 -1 2 2
Model M71 5 0 -3 -4 0 5 -4 -3 -3 -4 5 0 -4 -3 0 5	Model M72 10 0 -6 0 0 10 -8 5 -6 -8 -10 -4 0 5 -10 4	Model M73 2 1 -1 1 1 2 1 -1 -1 1 -2 2 -1 -2 2	Model M74 2 0 1 -1 0 2 1 -1 -1 1 -2 2 -1 -2 2	Model M75 4 1 2 -1 1 4 2 -4 2 2 4 -2 -1 -4 -2 4	Model M76 5 0 -3 -3 0 5 -4 -4 -3 -4 5 -5 -4 -5 5 -5
Model M77 2 0 1 0 0 2 1 -2 1 1 2 -1 -2 0 -2 1	Model M78 -1 -1 -2 -2 -1 4 -2 2 2 -2 4 4 -2 2 4 4	Model M79 5 0 4 0 0 5 3 5 4 3 5 3 0 -5 3 5	Model M80 2 -1 -2 -1 -1 2 1 2 -2 1 2 1 -1 2 1 2	Model M81 -2 -2 -1 -2 -2 2 1 2 -1 1 2 1 -2 2 1 2	Model M82 2 0 0 0 0 2 2 1 0 2 2 1 0 1 2 1
Model M83 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1	Model M84 1 0 0 0 0 1 1 1 0 1 1 1 0 1 1 1	Model M85 1 a b c a 1 d e b d 1 f c e f 1	Model M86 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1	Model M87 1 0 0 0 0 1 1 1 0 1 1 1 0 1 1 1	Model M88 1 a b c a 1 d e b d 1 f c e f 1

Petr Šimeček. Gaussian representation of independence models over four random variables.
In *COMPSTAT* conference, 2006.



Model M85



1	a	b	c
a	1	d	e
b	d	1	f
c	e	f	1

Where:

$$a = \frac{3}{632836} \sqrt{1107463},$$

$$b = 10c = \frac{100}{158209} \sqrt{1107463}$$

$$d = 10e = \frac{3}{4}, f = \frac{1}{10}$$



Šimeček's Question

Does every non-empty Gaussian CI model contain a rational point?

Model M85

Where:

$$a = \frac{3}{632836} \sqrt{1107463},$$

$$b = 10c = \frac{100}{158209} \sqrt{1107463}$$

$$d = 10e = \frac{3}{4}, f = \frac{1}{10}$$



1	a	b	c
a	1	d	e
b	d	1	f
c	e	f	1

$$\begin{pmatrix} 1 & -1/17 & -49/51 & -7/17 \\ -1/17 & 1 & 1/3 & 1/7 \\ -49/51 & 1/3 & 1 & 3/7 \\ -7/17 & 1/7 & 3/7 & 1 \end{pmatrix}$$



Complexity bounds

Let $f_1, \dots, f_r \in \mathbb{Z}[t_1, \dots, t_k]$ be integer polynomials in finitely many variables. We consider a system of polynomial constraints “ $f_i \bowtie 0$ ” where $\bowtie \in \{=, \neq, <, \leq, \geq, >\}$.

Theorem (Tarski's transfer principle)

If a polynomial system $\{f_i \bowtie 0\}$ has a solution over \mathbb{R} , then it has a solution in a finite real extension of \mathbb{Q} .

Theorem (Real Nullstellensatz)

A polynomial F vanishes on the semialgebraic set $\mathcal{K} = \{f_i \bowtie 0\}$ if and only if $F \in \sqrt[\mathbb{R}]{\mathcal{I}(f_i \bowtie 0)}$. The ideal $\mathcal{I}(f_i \bowtie 0)$, its real radical and the membership of F can be computed.

Keyword for this decision problem: “existential theory of the reals”.



Main results

Theorem

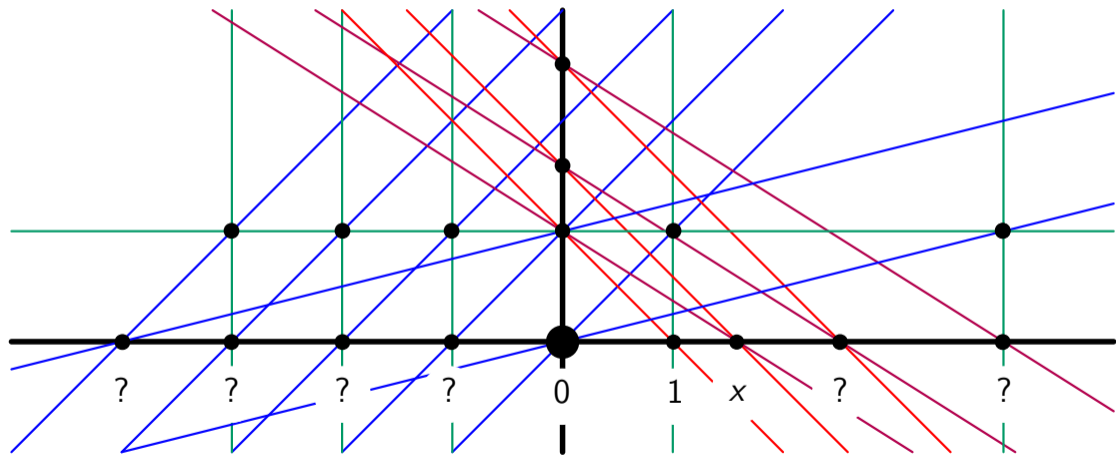
Let $d \geq 1$ and $\mathbb{Q}^{(d)}$ the field generated by all real algebraic numbers of degree at most d . For every d there exists a non-empty Gaussian CI model which has no $\mathbb{Q}^{(d)}$ -rational point.

Theorem

For every system of polynomials defining a semialgebraic set $\mathcal{K} = \{f_i \geq 0\}$ there exists a Gaussian CI model which is inhabited over \mathbb{R} if and only if \mathcal{K} is non-empty. Moreover, the description of this model is polynomially-sized in the description of \mathcal{K} .



Breakout riddle

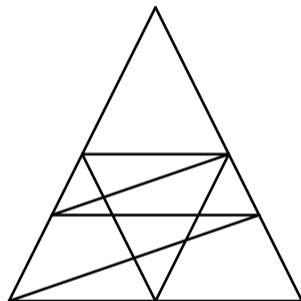


Algebra \subseteq Synthetic geometry

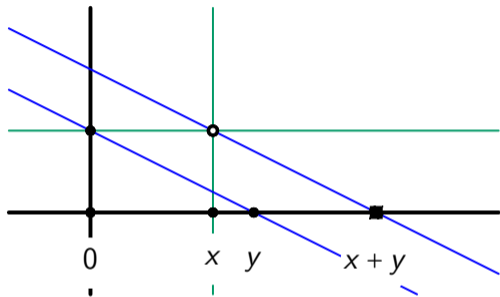
Point and line configuration for the equation $x^2 - 2 = 0$.

The configuration is specified by incidences between points and lines and also the parallelities of lines.

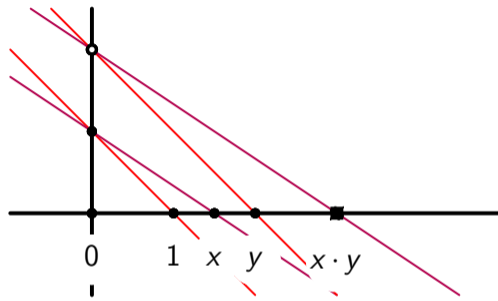
It is realizable over $\mathbb{Q}(\sqrt{2})$ but not over \mathbb{Q} .



Von Staudt constructions



Addition



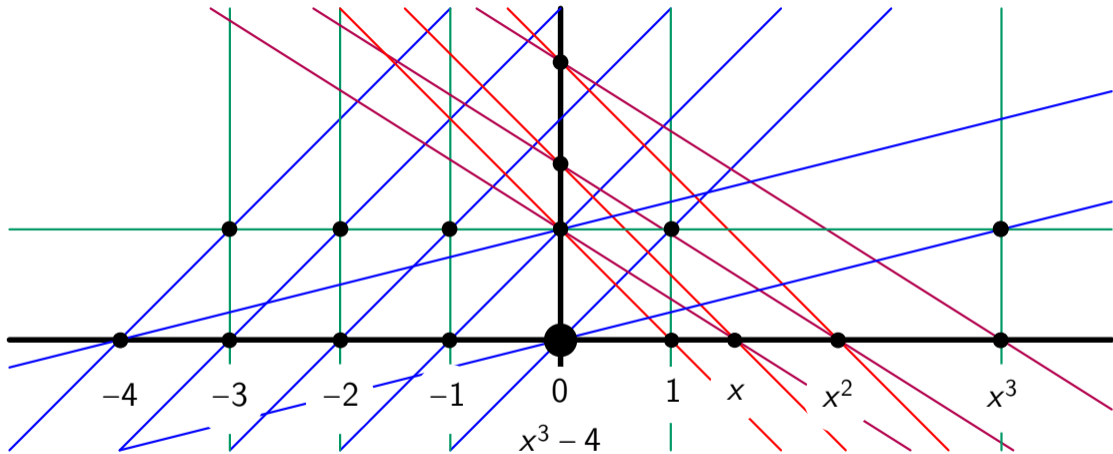
Multiplication

Lemma

The evaluation of integer polynomials can be encoded with incidence geometry.



The cube root of 4



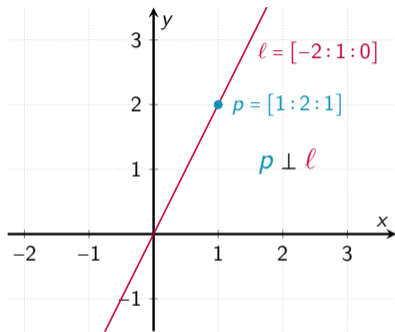
Incidence relations as conditional independence

Suppose $E = \{x, y, z\}$ and Σ_E is the identity matrix.

$$\Sigma[ij|E] = \Sigma[E] (\Sigma_{ij} - \Sigma_{i,E} \Sigma_E^{-1} \Sigma_{E,j}) \quad (\triangleleft)$$

$$\left. \begin{aligned} \Sigma[ij|E] = 0 &\Leftrightarrow \Sigma_{ij} = \langle \Sigma_{i,E}, \Sigma_{j,E} \rangle \\ \Sigma[ij] = 0 &\Leftrightarrow \Sigma_{ij} = 0 \end{aligned} \right\} \Leftrightarrow \Sigma_{i,E} \perp \Sigma_{j,E}$$

$p = \Sigma_{i,E} = [p_x : p_y : p_z]$ and $l = \Sigma_{j,E} = [l_x : l_y : l_z]$ are the *homogeneous coordinates* of a point and a line in the projective plane with $p \in l^\perp$.



Lemma

Incidence geometry can be encoded in CI constraints.



Condensed almost-principal minor

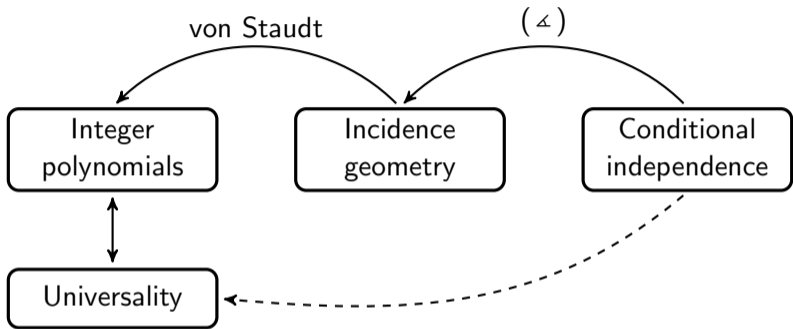
$$\begin{aligned}
 \Sigma[ij|xyz] &= x_{ij}x_{xx}x_{yy}x_{zz} + x_{iz}x_{jz}x_{xy}^2 - x_{iz}x_{jy}x_{xy}x_{xz} - x_{iy}x_{jz}x_{xy}x_{xz} + x_{iy}x_{jy}x_{xz}^2 \\
 &\quad - x_{iz}x_{jz}x_{xx}x_{yy} + x_{iz}x_{jx}x_{xz}x_{yy} + x_{ix}x_{jz}x_{xz}x_{yy} - x_{ij}x_{xz}^2x_{yy} \\
 &\quad + x_{iz}x_{jy}x_{xx}x_{yz} + x_{iy}x_{jz}x_{xx}x_{yz} - x_{iz}x_{jx}x_{xy}x_{yz} - x_{ix}x_{jz}x_{xy}x_{yz} \\
 &\quad - x_{iy}x_{jx}x_{xz}x_{yz} - x_{ix}x_{jy}x_{xz}x_{yz} + 2x_{ij}x_{xy}x_{xz}x_{yz} + x_{ix}x_{jx}x_{yz}^2 \\
 &\quad - x_{ij}x_{xx}x_{yz}^2 - x_{iy}x_{jy}x_{xx}x_{zz} + x_{iy}x_{jx}x_{xy}x_{zz} + x_{ix}x_{jy}x_{xy}x_{zz} \\
 &\quad - x_{ij}x_{xy}^2x_{zz} - x_{ix}x_{jx}x_{yy}x_{zz} \\
 &= x_{ij} - \sum_{k=x,y,z} x_{ik}x_{jk} = x_{ij} - \langle p_i, \ell_j \rangle.
 \end{aligned}$$



Polynomial evaluation as conditional independence

$$\begin{array}{c}
 p_1 \\
 \vdots \\
 p_n \\
 l_1 \\
 \vdots \\
 l_m \\
 x \\
 y \\
 z
 \end{array}
 \left(
 \begin{array}{ccc|ccc|ccc}
 p_1 & \dots & p_n & l_1 & \dots & l_m & x & y & z \\
 p_1^* & & \langle p, p' \rangle & & & & p_1^x & p_1^y & p_1^z \\
 & \ddots & & & \langle p, l \rangle & & & \vdots & \\
 \langle p', p \rangle & & p_n^* & & & & p_n^x & p_n^y & p_n^z \\
 \hline
 & & & l_1^* & & \langle l, l' \rangle & l_1^x & l_1^y & l_1^z \\
 & \langle l, p \rangle & & & \ddots & & & \vdots & \\
 \hline
 & & & \langle l', l \rangle & & l_m^* & l_m^x & l_m^y & l_m^z \\
 \hline
 p_1^x & & p_n^x & l_1^x & & l_m^x & 1 & 0 & 0 \\
 p_1^y & \dots & p_n^y & l_1^y & \dots & l_m^y & 0 & 1 & 0 \\
 p_1^z & & p_n^z & l_1^z & & l_m^z & 0 & 0 & 1
 \end{array}
 \right)$$





Theorem

To every polynomial system $\{f_i \approx 0\}$ there is a set of CI constraints which has a model over a field \mathbb{K}/\mathbb{Q} if and only if the polynomial system has a solution in \mathbb{K} .



Theorem

Let $d \geq 1$ and $\mathbb{Q}^{(d)}$ the field generated by all real algebraic numbers of degree at most d . For every d there exists a non-empty Gaussian CI model which has no $\mathbb{Q}^{(d)}$ -rational point.

Theorem

For every system of polynomials defining a semialgebraic set $\mathcal{K} = \{f_i \bowtie 0\}$ there exists a Gaussian CI model which is inhabited over \mathbb{R} if and only if \mathcal{K} is non-empty. Moreover, the description of this model is polynomially-sized in the description of \mathcal{K} .

Question: What is the smallest n (≥ 5) for which there is an n -variate Gaussian CI model without rational point?





Tobias Boege.

Incidence geometry in the projective plane via almost-principal minors of symmetric matrices, 2021.

[arXiv:2103.02589](https://arxiv.org/abs/2103.02589).



Jürgen Bokowski and Bernd Sturmfels.

Computational synthetic geometry, volume 1355 of *Lecture Notes in Mathematics*. Springer, 1989.



Jürgen Richter-Gebert.

Perspectives on projective geometry. A guided tour through real and complex geometry. Springer, 2011.



Petr Šimeček.

Gaussian representation of independence models over four random variables.
In *COMPSTAT conference*, 2006.



$\bowtie \in \{=\}$ suffices

In the real numbers, the solvability of a system of equations is just as hard as equations, inequations and inequalities if we introduce a new variable y :

- $f(x) \neq 0 \Leftrightarrow \exists y : yf(x) - 1 = 0$ ($f(x)$ has a multiplicative inverse)
- $f(x) \geq 0 \Leftrightarrow \exists y : f(x) - y^2 = 0$ ($f(x)$ is a square, i.e. non-negative)
- $f(x) > 0 \Leftrightarrow \exists y : y^2 f(x) - 1 = 0$ ($f(x)$ has an inverse which is a square)

