What is a semigraphoid?

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Conditional independence

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- For $i, j \in N$ and $K \subseteq N \setminus ij$ the conditional independence $\xi_i \perp\!\!\!\perp \xi_j \mid \xi_K$ (short: $(ij|K)$) holds if and only if $p_{ij|K}(x, y \mid z) = p_{i|K}(x \mid z) \cdot p_{j|K}(y \mid z)$ for all events x of ξ_i, y of ξ_j and almost every atomic event z of ξ_K.

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The following equivalences are true for the conditional independence structure:

$$\left.\begin{array}{lcl} (ij|L) \wedge (ik|jL) & \Leftrightarrow & (ij|kL) \wedge (ik|L), \\ (i|K) & \Leftrightarrow & (i|N \setminus i) \wedge \bigwedge_{\substack{j \in N \setminus iK, \\ K \subseteq L \subseteq N \setminus ij}} (ij|L) \end{array}\right\} \text{ semigraphoid axioms}$$

CI implication and algebraic statistics

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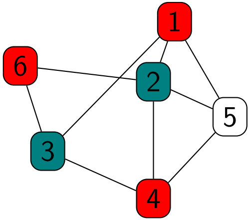
- For Gaussian random variables, the problem is solvable but $\exists\mathbb{R}$-complete.

Separation in graphs

- Let $G = (V, E)$ be an undirected graph and $(ij|K) \in \mathcal{A}_V$.



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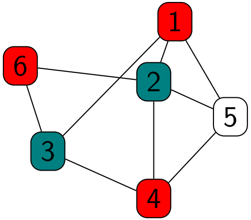

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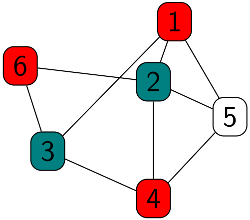
- The set $[\![G]\!] \coloneqq \{(ij|K) : K \text{ separates } i \text{ and } j\}$ is a semigraphoid which also satisfies $(ij|L) \Rightarrow (ij|kL)$ and other special properties.



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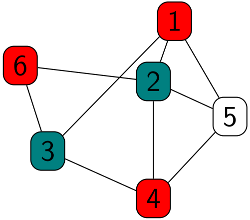
- The realizability problem is efficiently solvable. As statistical models, they have a rational parametrization, are smooth manifolds, etc.



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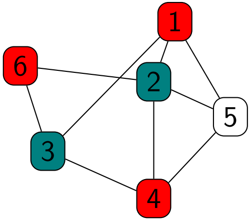

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Graphical models (of various types) are important in statistics, AI and applications in the sciences (as Markov random fields in physics, or phylogenetic trees in biology).

Modularity of submodular functions

- Let $f : 2^N \to \mathbb{R}$ be a locally submodular function, i.e., for each $(ij|K) \in \mathcal{A}_N$:

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Every semigraphoid has a "lattice of (cyclic) flats" attached. Nothing is known about additional properties of these lattices.

The language of semigraphoids: geometry $\leftrightarrow$ information theory

Condition on semigraphoid $\mathcal{L}$	Matroid concept	Information theory concept
$(i\|\varnothing) \in \mathcal{L}$	loop	constant random variable
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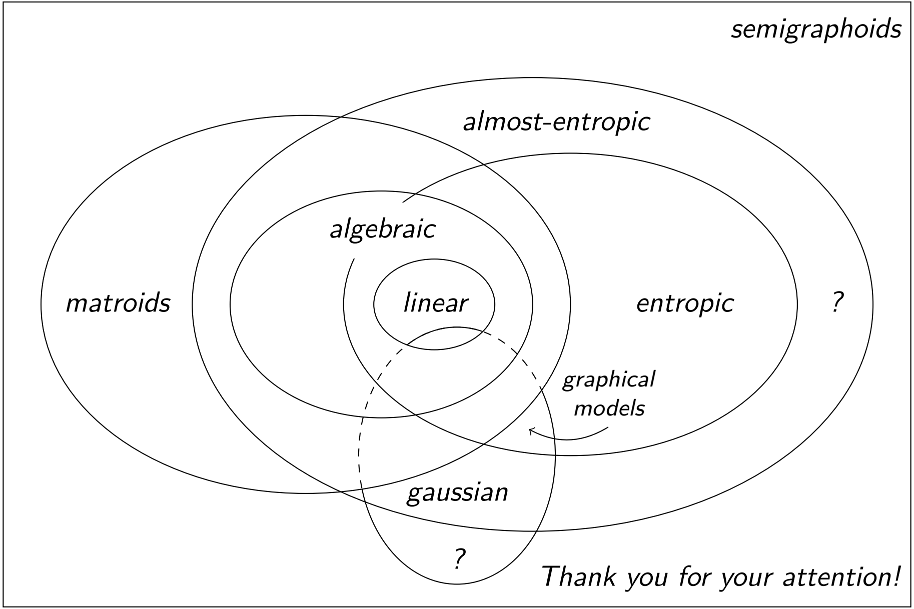
Every semigraphoid has a "Tutte polynomial" via deletion-contraction recurrence.
Nobody knows what it does.



Thank you for your attention!