Causality and independence in stationary diffusions

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Algebraic methods in dynamics and particle physics
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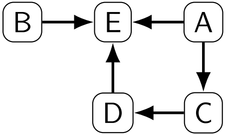

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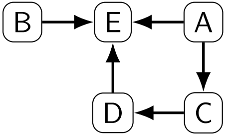

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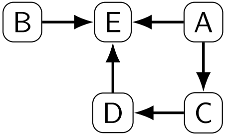
- The causal diagram may come from expert knowledge or it might be learned from observational data.



Causal modeling in physics

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- Časlav Brukner. "Quantum causality". In: *Nature Physics* 10.4 (2014). DOI: 10.1038/nphys2930.
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Linear structural equation models

- Linear structural equation models provide a simple encoding of causal diagrams $\mathcal{G}$. Define a random variable X_j as a linear function of its parents, up to some noise:

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- In short: $X = \Lambda X + \varepsilon$ where Λ is $\mathcal{G}$-sparse and $\varepsilon \sim \mathcal{N}(0, \Omega)$ with Ω diagonal.
- Solutions to this system are multivariate normal distributions with zero mean and covariance matrix Σ satisfying the congruence

$$\boxed{(I - \Lambda)^{\mathsf{T}} \Sigma (I - \Lambda) = \Omega.}$$

LSEMs as statistical models

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⋝ Interactions between nodes only through the prescribed causal mechanism! ⋜

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Some good news: Markov equivalence can be characterized combinatorially!

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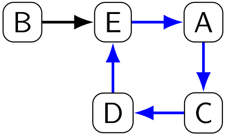

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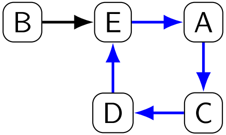

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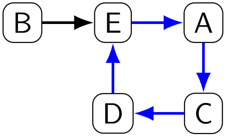

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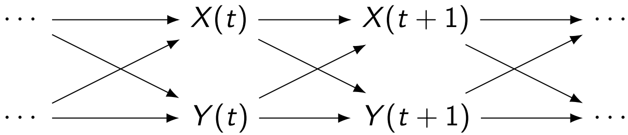

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$$
\begin{tikzcd}
\mathrm{B} \arrow[r] & \mathrm{E} \arrow[r] & \mathrm{A} \arrow[d] \\
 & \mathrm{D} \arrow[u] & \mathrm{C} \arrow[l]
\end{tikzcd}
$$

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\begin{tikzcd}
\cdots \arrow[r] \arrow[dr] & X(t) \arrow[r] \arrow[dr] & X(t+1) \arrow[r] \arrow[dr] & \cdots \\
\cdots \arrow[r] \arrow[ur] & Y(t) \arrow[r] \arrow[ur] & Y(t+1) \arrow[r] \arrow[ur] & \cdots
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- Formally, take the stationary distribution of the Ornstein–Uhlenbeck process

$$\mathrm{d}\mathbb{X}(t) = M\mathbb{X}(t)\mathrm{d}t + D\mathrm{d}\mathbb{W}(t),$$

where M is $\mathcal{G}$-sparse, $\mathbb{W}$ a standard Brownian motion, and D diagonal.

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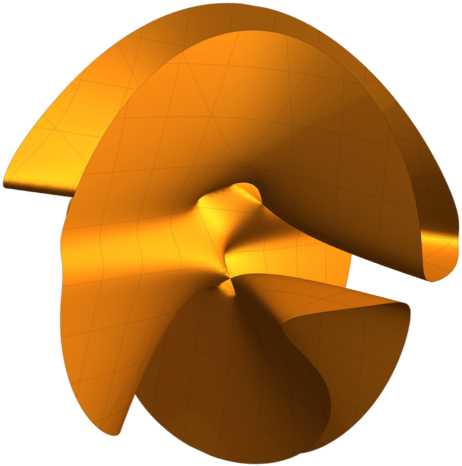
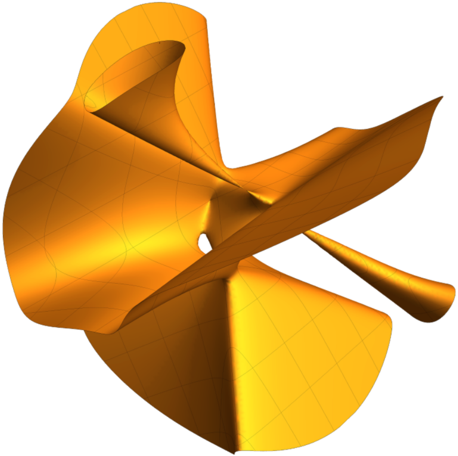
$\geqslant$ The Lyapunov model is an irreducible algebraic variety! $\leqslant$

Beauty is in the eye of the beholder...





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Lyapunov models are more often globally identifiable than LSEMs!

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Strange: Lyapunov models are **not** defined by conditional independence relations.

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Corollary

Model equivalence can be checked in polynomial time. Model equivalence classes can be enumerated in an output-sensitive manner.

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Corollary

Lyapunov models refine the Markov equivalence classes of Bayesian networks.

A sparsest (connected) example of non-identifiability

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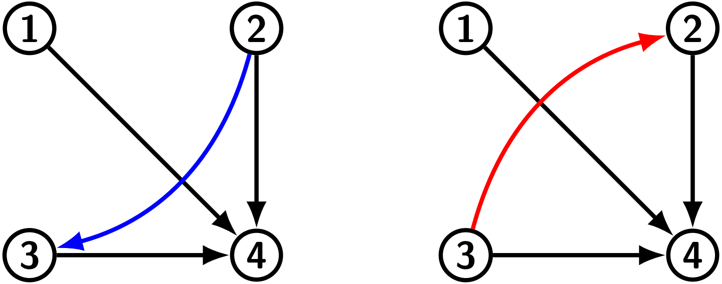

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- Despite that, they encode strictly more causal information than Bayesian networks.

		Lyapunov DAG models		Bayesian network models	
n	DAGs	Distinct models	Identifiable	Distinct models	Identifiable
3	25	17	13	11	4
4	543	461	423	185	59
5	29 281	27 697	26 761	8 782	2 616
6	3 781 503	3 715 745	3 665 673	1 067 825	306 117

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