

Universality of Gaussian conditional independence models

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Workshop on Algorithmic aspects of information theory
Schloss Dagstuhl, 26 July 2022

Basic questions

Definition

A *CI constraint* is a CI statement $[\xi_i \perp\!\!\!\perp \xi_j \mid \xi_K]$ or its negation $\neg[\xi_i \perp\!\!\!\perp \xi_j \mid \xi_K]$ constraining a random vector ξ .

- ▶ How hard is it to decide if a set of constraints is consistent?
- ▶ How hard is it to *certify* consistency by exhibiting a distribution?
- ▶ What is the geometric structure of the models?

Conjectures

Studený's Question (2005)

If a set of CI constraints is satisfiable by regular Gaussian random variables, then is it satisfiable by discrete random variables?

Šimeček's Question (2006)

If a set of CI constraints is satisfiable by regular Gaussian random variables, then can the covariances be chosen rational?

Model M1 4 0 2 -2 0 4 1 -3 2 1 4 -3 -2 -3 -3 4	Model M2 4 1 -3 2 1 4 -3 2 -3 -4 4 -3 2 -3 4	Model M3 4 -3 -3 -3 -3 4 3 2 -3 4 1 -3 2 1 4	Model M4 4 0 1 -1 0 4 1 1 1 1 4 1 -1 1 1 4	Model M5 4 0 2 -2 0 4 2 -3 2 2 4 -3 -2 -3 -3 4	Model M6 4 0 3 -2 0 4 2 -3 3 2 4 -3 -2 -3 -3 4
Model M7 4 0 -2 2 0 4 -3 1 -2 -3 -4 1 2 1 -1 4	Model M8 4 1 -2 2 1 4 -3 2 -2 -3 4 -3 2 -2 1 4	Model M9 4 2 -3 3 2 4 -3 1 -3 -3 4 -2 3 1 -2 4	Model M10 4 1 -2 2 1 4 -3 2 -2 -3 4 -2 2 2 -2 4	Model M11 4 0 -3 -3 0 4 1 2 -3 1 4 2 -3 2 2 4	Model M12 4 0 -2 1 0 4 -3 -2 -2 -3 4 0 1 -2 0 4
Model M13 4 -1 -2 -2 -1 4 -2 2 2 -2 4 -3 -2 -2 -3 4	Model M14 4 1 -2 2 1 4 -1 -2 2 1 4 -2 -2 -2 2 4	Model M15 6 1 4 2 1 6 5 3 -4 -5 6 -3 2 3 -3 6	Model M16 4 0 2 -1 0 4 2 -3 2 2 4 -3 -1 -3 -3 4	Model M17 4 0 -2 1 0 4 -3 2 -2 -3 4 -2 1 2 -2 4	Model M18 4 0 -2 2 0 4 -3 -1 -2 -3 4 -1 2 1 -1 4
Model M19 4 0 2 -2 0 4 -1 -3 2 1 -4 0 -2 -3 0 4	Model M20 4 0 -2 -3 0 4 -2 -1 -2 2 4 1 -3 -1 1 4	Model M21 4 0 -2 2 0 4 -2 2 -2 -2 4 -1 2 2 -1 4	Model M22 4 0 2 -2 0 4 -2 2 -2 -2 4 -1 2 2 -1 4	Model M23 4 0 2 -1 0 4 1 -2 2 1 4 -2 -1 -2 2 4	Model M24 6 0 -3 1 0 6 -4 3 -3 -4 6 -2 1 3 -2 6
Model M25 8 0 4 -3 0 8 4 -7 4 4 8 -6 -3 -7 -6 8	Model M26 8 -3 -6 -6 -3 8 4 4 -6 4 8 2 -6 -4 2 8	Model M27 8 0 4 2 0 8 2 -5 4 2 8 4 -2 -5 -4 8	Model M28 10 4 -5 -8 -4 10 2 5 -5 2 10 1 -8 -5 1 10	Model M29 10 4 -8 -5 -4 10 2 8 -8 2 10 4 -8 2 10 4	Model M30 4 0 -2 2 0 4 -1 -1 -2 1 4 0 -2 -1 0 4
Model M31 12 -3 -5 -2 -3 12 6 8 -6 6 12 8 -2 8 8 12	Model M32 20 5 -10 10 5 20 -10 10 -10 -10 20 -8 10 10 -8 20	Model M33 4 0 -3 -3 0 4 0 1 -3 0 4 1 -3 1 1 4	Model M34 4 -1 -3 -2 -1 4 1 2 -3 1 4 2 -2 2 2 4	Model M35 4 0 -2 -3 0 4 -3 -2 -2 -3 4 -2 -3 -2 0 4	Model M36 4 1 -2 2 1 4 -2 2 -2 -2 4 -1 2 2 -1 4
Model M37 10 0 6 -3 0 10 4 -8 6 4 10 -5 -3 -8 -5 10	Model M38 4 0 -2 0 0 4 -3 3 -2 -3 4 -3 0 3 -4 3	Model M39 4 0 1 -2 0 4 0 -3 1 0 4 -2 -2 -3 -2 4	Model M40 4 0 -2 1 0 4 -2 1 -2 -2 4 -2 1 1 -2 4	Model M41 12 3 -9 6 3 12 -9 6 -9 9 12 -8 6 6 -8 12	Model M42 4 0 -2 0 0 4 -3 0 -2 -3 4 -1 0 0 -1 4
Model M43 4 1 -2 1 1 4 -2 1 -2 1 4 -2 1 1 -2 4	Model M44 4 0 -2 -1 0 4 -3 0 -2 1 4 -2 -1 -3 -2 4	Model M45 4 0 -2 -3 0 4 -0 -1 -2 1 4 -2 -3 -1 0 4	Model M46 6 2 -3 -4 2 6 -1 -3 -3 -1 6 2 -4 -3 2 6	Model M47 4 0 0 0 0 4 -1 -3 -1 -1 4 0 0 -3 -1 4	Model M48 4 0 0 0 0 4 -0 -3 -1 4 0 0 0 -3 -2 4

Model M49 4 0 0 0 0 4 1 -2 0 1 4 -2 0 -2 -2 4	Model M50 4 0 -3 0 0 4 0 1 -3 0 4 0 0 1 0 4	Model M51 4 0 0 0 0 4 0 -3 0 0 4 0 0 -3 0 4	Model M52 1 0 0 0 0 1 0 0 0 0 1 0 0 0 0 1	Model M53 4 -3 -3 -3 -3 4 1 1 -3 4 1 1 -3 1 4 3	
Model M54	Model M55	Model M56	Model M57	Model M58	
Model M59 4 -3 -2 1 -3 4 -2 -2 -2 2 4 1 1 -2 2 4	Model M60 2 0 1 -1 0 2 -1 -1 1 -1 2 -1 -1 -1 -1 2	Model M61 5 0 4 -3 0 5 2 -4 4 2 5 -4 -3 -4 -4 5	Model M62 8 -2 -4 -7 -2 8 -4 -2 -4 8 2 -7 -7 -2 2 8	Model M63 10 0 4 -6 0 10 -3 -8 -4 -3 10 0 -6 -8 0 10	Model M64 4 1 1 -2 1 4 -2 -2 -2 -2 4 -2 -2 -2 2 4
Model M65 9 0 -6 -6 0 9 3 -3 -6 3 9 -1 -6 -3 -1 9	Model M66 8 0 -6 3 0 8 8 -6 -6 8 8 -6 3 5 -6 8	Model M67 14 -13 -11 -7 -13 14 7 2 -11 7 14 13 -7 2 13 14	Model M68 10 0 -6 -3 0 10 -8 -4 -6 -8 10 -5 3 -4 -5 10	Model M69 25 9 20 -15 9 25 15 -20 20 15 25 -24 -15 -20 -24 25	Model M70 2 1 1 -1 1 2 1 -1 1 1 2 -1 -1 -1 -2 2
Model M71 5 0 -3 -3 -3 4 5 0 -4 3 0 5 -4 3 0 5	Model M72 10 0 -6 0 0 10 -8 5 -8 -8 10 -4 0 5 -4 10	Model M73 2 1 -1 -1 1 2 1 -1 -1 1 2 -2 -1 -1 -2 2	Model M74 2 0 1 -1 0 2 -1 -1 -1 1 2 -2 -1 -1 -2 2	Model M75 4 1 2 -1 1 4 -2 -4 -2 2 4 -2 -1 -4 -2 4	Model M76 5 0 3 -3 0 5 -4 4 -4 3 -5 5 -2 -4 -5 5
Model M77 2 0 1 0 0 2 1 -2 1 1 2 -1 0 -2 1 2	Model M78 4 -1 2 -2 -1 4 -2 -2 2 -2 4 -4 -2 2 -4 4	Model M79 5 0 4 0 0 5 3 -5 4 3 5 -3 0 -5 -3 5	Model M80 2 -1 -2 -1 -1 2 1 2 -2 1 2 1 -1 2 1 2	Model M81 2 -2 -1 -2 -2 2 1 2 -1 1 2 1 -2 2 1 2	Model M82 2 0 0 0 0 2 -1 2 -2 2 1 2 0 -2 1 2
Model M83 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1	Model M84 1 0 0 0 0 1 1 1 0 1 1 1 0 1 1 1	Model M85 1 a b c a 1 d e b d 1 f c e f 1	$p_{\text{H0}} = \frac{3}{\sqrt{10743}}$ $a = 10c = \frac{300}{\sqrt{10743}}$ $b = 10e = \frac{300}{\sqrt{10743}}$ $d = 10f = \frac{3}{10}$		
Model M86	Model M87	Model M88			

Petr Šimeček. "Gaussian representation of independence models over four random variables".
In: *COMPSTAT conference*. 2006

Gaussian conditional independence

Assume $\xi = (\xi_i : i \in N)$ are jointly Gaussian with covariance matrix $\Sigma \in \text{PD}_N$.

Definition

The polynomial $\Sigma[K] := \det \Sigma_{K,K}$ is a *principal minor* of Σ and $\Sigma[ij | K] := \det \Sigma_{iK,jK}$ is an *almost-principal minor*.

- ▶ Σ is PD if and only if $\Sigma[K] > 0$ for all $K \subseteq N$.
- ▶ $[\xi_i \perp\!\!\!\perp \xi_j \mid \xi_K]$ holds if and only if $\Sigma[ij | K] = 0$.
- ▶ $\mathbb{E}[\xi] = \mu$ is irrelevant.

Very special polynomials

$$\Sigma[ij |] = x_{ij}$$

$$\Sigma[ij | k] = x_{ij}x_{kk} - x_{ik}x_{jk}$$

$$\Sigma[ij | kl] = x_{ij}x_{kk}x_{ll} - x_{il}x_{jl}x_{kk} + x_{il}x_{jk}x_{kl} + x_{ik}x_{jl}x_{kl} - x_{ij}x_{kl}^2 - x_{ik}x_{jk}x_{ll}$$

$$\begin{aligned}\Sigma[ij | klm] = & x_{ij}x_{kk}x_{ll}x_{mm} + x_{im}x_{jm}x_{kl}^2 - x_{im}x_{jl}x_{kl}x_{km} - x_{il}x_{jm}x_{kl}x_{km} + \\ & x_{il}x_{jl}x_{km}^2 - x_{im}x_{jm}x_{kk}x_{ll} + x_{im}x_{jk}x_{km}x_{ll} + x_{ik}x_{jm}x_{km}x_{ll} - \\ & x_{ij}x_{km}^2x_{ll} + x_{im}x_{jl}x_{kk}x_{lm} + x_{il}x_{jm}x_{kk}x_{lm} - x_{im}x_{jk}x_{kl}x_{lm} - \\ & x_{ik}x_{jm}x_{kl}x_{lm} - x_{il}x_{jk}x_{km}x_{lm} - x_{ik}x_{jl}x_{km}x_{lm} + 2x_{ij}x_{kl}x_{km}x_{lm} + \\ & x_{ik}x_{jk}x_{lm}^2 - x_{ij}x_{kk}x_{lm}^2 - x_{il}x_{jl}x_{kk}x_{mm} + x_{il}x_{jk}x_{kl}x_{mm} + \\ & x_{ik}x_{jl}x_{kl}x_{mm} - x_{ij}x_{kl}^2x_{mm} - x_{ik}x_{jk}x_{ll}x_{mm}\end{aligned}$$

⋮

Gaussian CI models

Definition

The *model* of a set of CI constraints is the set of all PD matrices which satisfy them.

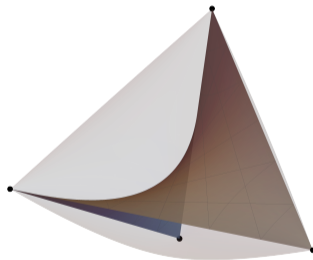


Figure: Model of $\Sigma[12|3] = a - bc = 0$ in the space of 3×3 correlation matrices.

Models and inference

Consider two sets of CI statements \mathcal{P} and \mathcal{Q} :

$$\bigwedge \mathcal{P} \Rightarrow \bigvee \mathcal{Q}$$

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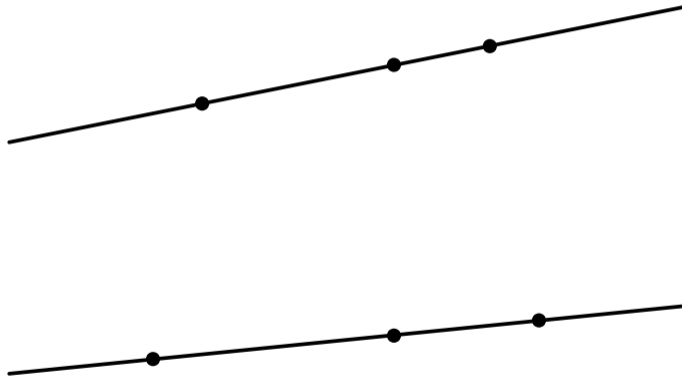
Models and inference

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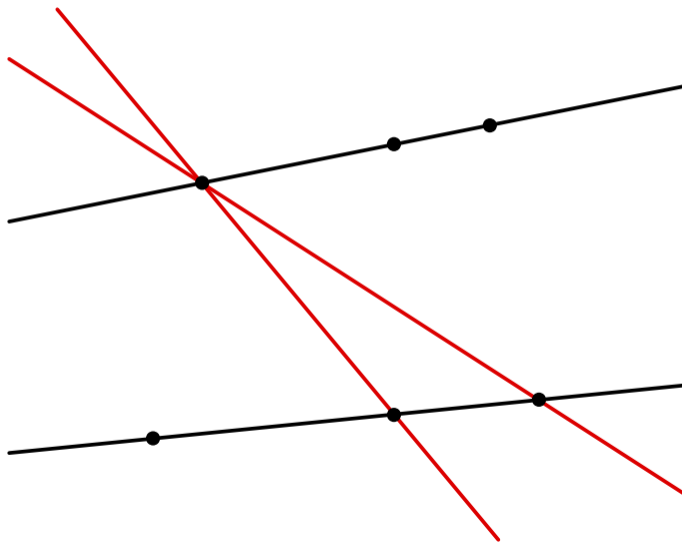
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Reasoning about CI statements in normally distributed random variables is **the same** as reasoning about the vanishing of very special kinds of determinants on very special kinds of varieties inside the positive definite matrices.

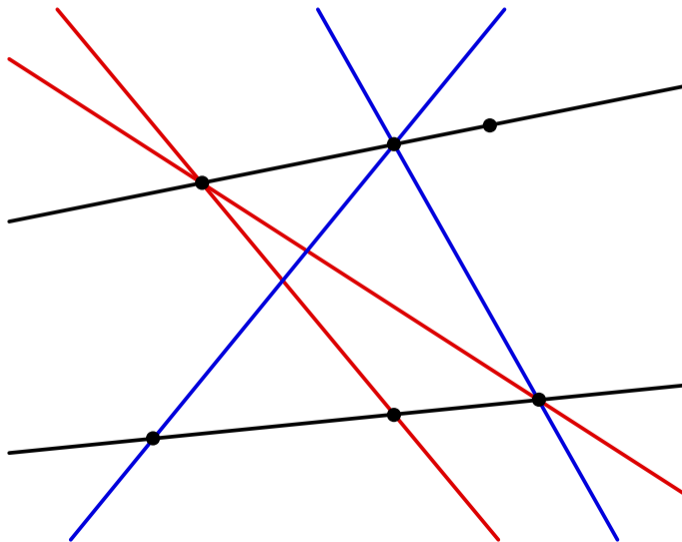
For ancient geometers: conditional independence \approx collinearity



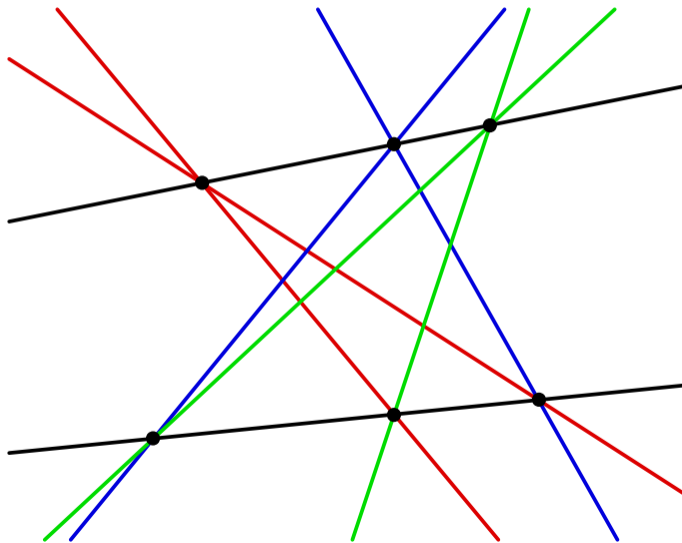
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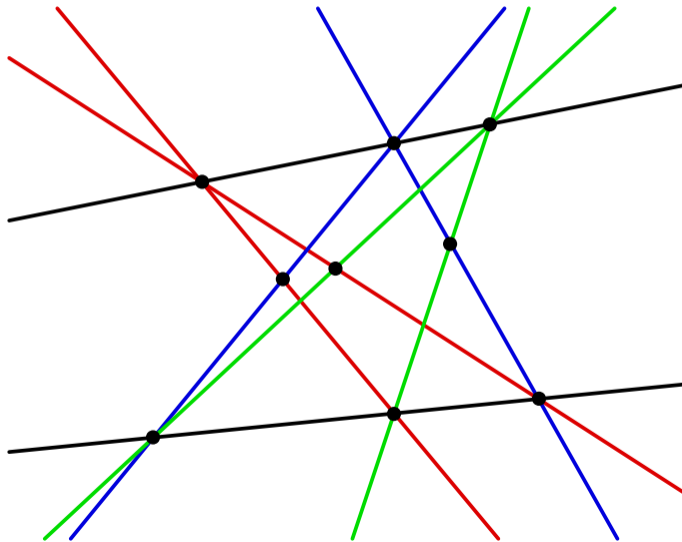
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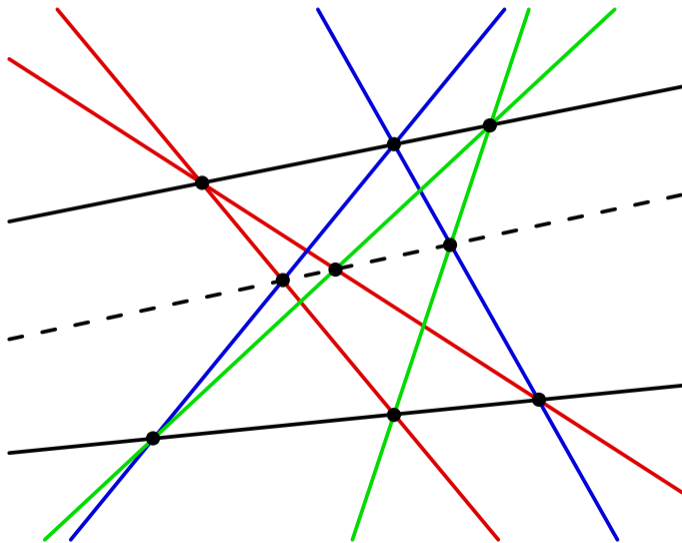
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Normal form for proofs and refutations

Let $f_i \in \mathbb{Z}[t_1, \dots, t_k]$ be integer polynomials in finitely many variables.

Theorem (Tarski's transfer principle)

If a polynomial system $\{f_i \bowtie_i 0\}$, where $\bowtie_i \in \{=, \neq, <, \leq, \geq, >\}$, has a solution over \mathbb{R} , then it has a solution in a finite real extension of \mathbb{Q} .

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→ If $\bigwedge \mathcal{P} \Rightarrow \bigvee \mathcal{Q}$ is false, there exists a counterexample matrix Σ with algebraic entries.

$[12 |] \wedge [12 | 3] \Rightarrow [13 |]$ is false and a counterexample is

$$\begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1 \end{pmatrix}.$$

Normal form for proofs and refutations

Let $f_i, g_j, h_k \in \mathbb{Z}[t_1, \dots, t_k]$ be integer polynomials in finitely many variables.

Theorem (Positivstellensatz)

A polynomial system $\{f_i = 0, g_j \geq 0, h_k \neq 0\}$ is infeasible if and only if there exist $f \in \text{ideal}(f_i)$, $g \in \text{cone}(g_j)$ and $h \in \text{monoid}(h_k)$ such that $g + h^2 = f$.

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→ If $\bigwedge \mathcal{P} \Rightarrow \bigvee \mathcal{Q}$ is **true**, there exists an algebraic proof for it with integer coefficients.

$[12|] \wedge [12|3] \Rightarrow [13|] \vee [23|]$ is true and a proof is the **final polynomial**

$$\Sigma[13|] \cdot \Sigma[23|] = \Sigma[3] \cdot \Sigma[12|] - \Sigma[12|3].$$

Computer algebra proves laws of probabilistic reasoning

The following inference rule is valid for all positive definite 5×5 matrices:

$$[12|] \wedge [14|5] \wedge [23|5] \wedge [35|1] \wedge [45|2] \wedge [15|23] \wedge [34|12] \wedge [24|135] \Rightarrow [25|] \vee [34|].$$

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$$\begin{aligned} & [25 |] [34 |] \cdot [1] [2] [3] [15] = \\ & (cd^2egr + bd^2fgr - ad^2grh - 2cd^2e^2i - 2bd^2efi - 2pdfgri + 2ad^2ehi + 2pdefi^2 - 2pdqhi^2 + 2pcqi^3 + \\ & 2pdqrij - 2pbqi^2j - pcegrt + pbfgrt + pagrht + 2pce^2it - 2pcqrit + 2pbqhit - 2paehit) \cdot [12 |] + \\ & (pdqer + pbqgr - 2pbqei) \cdot [14 | 5] - (pcdqr + p^2fgr - 2pbcqi + 2pb^2qj - 2p^2qrj) \cdot [23 | 5] + \\ & (cdqgr - 2cdqei + 2pqghi - 2pqfi^2 - pqgrj + 2pqueij - 2pe^2ft + 2pqfrit) \cdot [35 | 1] + \\ & (pd^2er - 2pbdei + p^2gri + 2pb^2et - 2p^2ert) \cdot [45 | 2] - (2pdfi - 2pbft) \cdot [15 | 23] - \\ & (d^2gr - 2d^2ei - pgrt + 2peit) \cdot [34 | 12] - 2pqi \cdot [24 | 135]. \end{aligned}$$

Computer algebra proves laws of probabilistic reasoning

```
R = QQ[p,a,b,c,d, q,e,f,g, r,h,i, s,j, t];
X = genericSymmetricMatrix(R,p,5);
I = ideal(
  det X_{0}^{1}, det X_{0,3}^{2,3}, det X_{0,4}^{3,4},
  det X_{1,4}^{2,4}, det X_{2,0}^{4,0}, det X_{3,1}^{4,1},
  det X_{0,1,2}^{4,1,2}, det X_{2,0,1}^{3,0,1},
  det X_{1,0,2,4}^{3,0,2,4}
);
U = g*h*p*q*r*(p*t-d^2); -- [25|][34|] · [1][2][3][15] ∈ monoid(V)
U % I --> 0, meaning monoid(V) ∩ ideal(V) ≠ ∅ in Q[X]
-- Get a proof that U is in I:
G = gens I; -- the equations generating ideal(V)
H = U // G; -- linear combinators for U from G
U == G*H --> true
```

Consistency checking is hard

The complexity class $\exists\mathbb{R}$ contains all decision problems which can be reduced in polynomial time to the feasibility of a semialgebraic set:

- ▶ polynomial optimization
- ▶ computational geometry
- ▶ algebraic statistics . . .

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Theorem

The problem of deciding whether a general CI model is non-empty is complete for $\exists\mathbb{R}$.

Consistency certification is hard

Šimeček's Question (2006)

Does every non-empty Gaussian CI model contain a rational point?

Or: can every wrong inference rule be refuted over \mathbb{Q} ?


Consistency certification is hard

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Model M85 *Where:*



$a = \frac{3}{632836} \sqrt{1107463},$
 $b = 10c = \frac{100}{158209} \sqrt{1107463}$
 $d = 10e = \frac{3}{4}, f = \frac{1}{10}$

$$\begin{pmatrix} 1 & -1/17 & -49/51 & -7/17 \\ -1/17 & 1 & 1/3 & 1/7 \\ -49/51 & 1/3 & 1 & 3/7 \\ -7/17 & 1/7 & 3/7 & 1 \end{pmatrix}$$

Consistency certification is hard

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Theorem

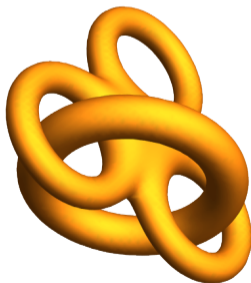
For every finite real extension \mathbb{K} of \mathbb{Q} there exists a CI model \mathcal{M} such that $\mathcal{M} \neq \emptyset$ but $\mathcal{M} \cap \text{PD}_N(\mathbb{K}) = \emptyset$.

Model topology can be bad

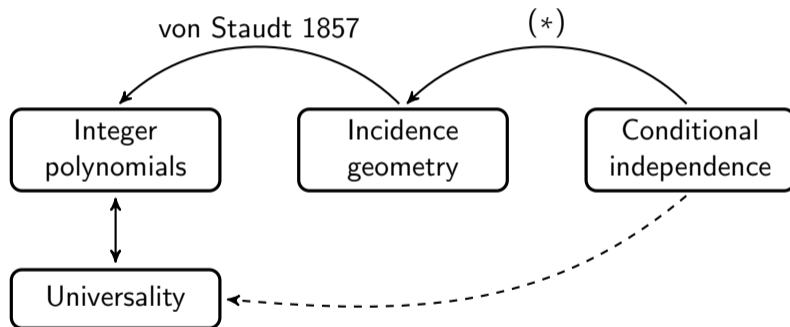
An **oriented** CI model is specified by **sign constraints** on partial correlations.

Theorem

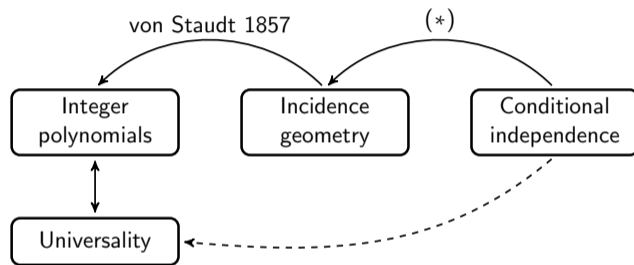
For every primary basic semialgebraic set Z there exists an oriented CI model \mathcal{M} which is homotopy-equivalent to Z .



Universality theorems







Universality theorems

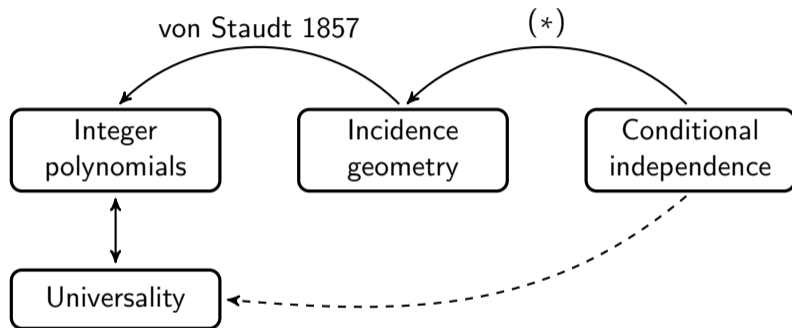


- ▶ Realization spaces of rank-3 matroids
- ▶ Realization spaces of 4-polytopes
- ▶ Nash equilibria of 3-person games
- ▶ Gaussian CI models with conditioning sets of size up to 3 ...

References

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-  Jürgen Bokowski and Bernd Sturmfels. *Computational synthetic geometry*. Vol. 1355. Lecture Notes in Mathematics. Springer, 1989.
-  Jürgen Richter-Gebert. *Perspectives on projective geometry. A guided tour through real and complex geometry*. Springer, 2011, pp. xxii + 571.
-  Petr Šimeček. “Gaussian representation of independence models over four random variables”. In: *COMPSTAT conference*. 2006.

Universality theorems: Background



Theorem

To every polynomial system $\{f_i \neq 0\}$ there is a set of CI constraints which has a model over a field \mathbb{K}/\mathbb{Q} if and only if the polynomial system has a solution in \mathbb{K} .

Very special polynomials

$\Sigma[ij|] = x_{ij} \rightarrow$ impose $x_{kl} = x_{km} = x_{lm} = 0$ on a correlation matrix, then:

$$\begin{aligned} \Sigma[ij|klm] &= x_{ij}x_{kk}x_{ll}x_{mm} + x_{im}x_{jm}x_{kl}^2 - x_{im}x_{jl}x_{kl}x_{km} - x_{il}x_{jm}x_{kl}x_{km} + x_{il}x_{jl}x_{km}^2 \\ &\quad - x_{im}x_{jm}x_{kk}x_{ll} + x_{im}x_{jk}x_{km}x_{ll} + x_{ik}x_{jm}x_{km}x_{ll} - x_{ij}x_{km}^2x_{ll} \\ &\quad + x_{im}x_{jl}x_{kk}x_{lm} + x_{il}x_{jm}x_{kk}x_{lm} - x_{im}x_{jk}x_{kl}x_{lm} - x_{ik}x_{jm}x_{kl}x_{lm} \\ &\quad - x_{il}x_{jk}x_{km}x_{lm} - x_{ik}x_{jl}x_{km}x_{lm} + 2x_{ij}x_{kl}x_{km}x_{lm} + x_{ik}x_{jk}x_{lm}^2 \\ &\quad - x_{ij}x_{kk}x_{lm}^2 - x_{il}x_{jl}x_{kk}x_{mm} + x_{il}x_{jk}x_{kl}x_{mm} + x_{ik}x_{jl}x_{kl}x_{mm} \\ &\quad - x_{ij}x_{kl}^2x_{mm} - x_{ik}x_{jk}x_{ll}x_{mm} \\ &= x_{ij} - \sum_{k=l,m} x_{ik}x_{jk} = x_{ij} - \left\langle \left(\begin{matrix} x_{ik} \\ x_{il} \\ x_{im} \end{matrix} \right), \left(\begin{matrix} x_{jk} \\ x_{jl} \\ x_{jm} \end{matrix} \right) \right\rangle. \end{aligned}$$

The rest is 19th century projective geometry. Keyword: *von Staudt constructions*.

Covariance matrix simulating a projective plane

$$\begin{array}{c}
 p_1 \\
 \vdots \\
 p_n \\
 l_1 \\
 \vdots \\
 l_m \\
 x \\
 y \\
 z
 \end{array}
 \left(
 \begin{array}{ccc|ccc|ccc}
 p_1 & \dots & p_n & l_1 & \dots & l_m & x & y & z \\
 p_1^* & & \langle p, p' \rangle & & & & p_1^x & p_1^y & p_1^z \\
 \vdots & \ddots & & & \langle p, l \rangle & & \vdots & & \\
 \langle p', p \rangle & & p_n^* & & & & p_n^x & p_n^y & p_n^z \\
 \hline
 & & & l_1^* & & \langle l, l' \rangle & l_1^x & l_1^y & l_1^z \\
 & \langle l, p \rangle & & & \ddots & & \vdots & & \\
 \hline
 & & & \langle l', l \rangle & & l_m^* & l_m^x & l_m^y & l_m^z \\
 \hline
 p_1^x & & p_n^x & l_1^x & & l_m^x & x^* & 0 & 0 \\
 p_1^y & \dots & p_n^y & l_1^y & \dots & l_m^y & 0 & y^* & 0 \\
 p_1^z & & p_n^z & l_1^z & & l_m^z & 0 & 0 & z^*
 \end{array}
 \right)$$