Universality of Gaussian conditional independence models

Tobias Boege

Max-Planck Institute for Mathematics in the Sciences, Leipzig

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Basic questions

Definition

A *CI constraint* is a CI statement $[\xi_i \perp\!\!\!\perp \xi_j \mid \xi_K]$ or its negation $\neg[\xi_i \perp\!\!\!\perp \xi_j \mid \xi_K]$ constraining a random vector ξ.

- How hard is it to decide if a set of constraints is consistent?
- How hard is it to *certify* consistency by exhibiting a distribution?
- What is the geometric structure of the models?

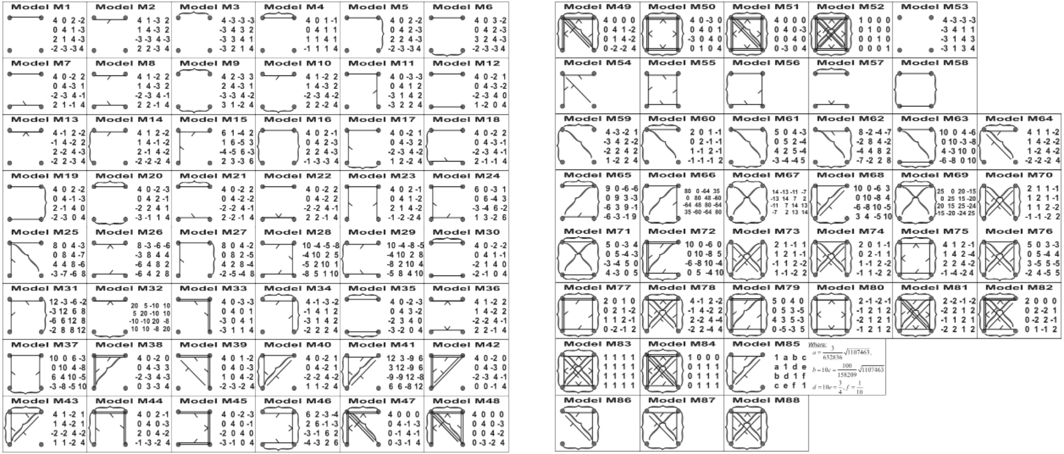
Conjectures

Studený's Question (2005)

If a set of CI constraints is satisfiable by regular Gaussian random variables, then is it satisfiable by discrete random variables?

Šimeček's Question (2006)

If a set of CI constraints is satisfiable by regular Gaussian random variables, then can the covariances be chosen rational?



Petr Šimeček. “Gaussian representation of independence models over four random variables”. In: *COMPSTAT conference*. 2006

Gaussian conditional independence

Assume $\xi = (\xi_i : i \in N)$ are jointly Gaussian with covariance matrix $\Sigma \in \mathrm{PD}_N$.

Definition

The polynomial $\Sigma[K] \coloneqq \det \Sigma_{K,K}$ is a *principal minor* of Σ and $\Sigma[ij \mid K] \coloneqq \det \Sigma_{iK,jK}$ is an *almost-principal minor*.

- Σ is PD if and only if $\Sigma[K] > 0$ for all $K \subseteq N$.
- $[\xi_i \perp\!\!\!\perp \xi_j \mid \xi_K]$ holds if and only if $\Sigma[ij \mid K] = 0$.
- $\mathbb{E}[\xi] = \mu$ is irrelevant.

Very special polynomials

$$
\begin{aligned}
\Sigma[ij \mid] &= x_{ij} \\
\Sigma[ij \mid k] &= x_{ij}x_{kk} - x_{ik}x_{jk} \\
\Sigma[ij \mid kl] &= x_{ij}x_{kk}x_{ll} - x_{il}x_{jl}x_{kk} + x_{il}x_{jk}x_{kl} + x_{ik}x_{jl}x_{kl} - x_{ij}x_{kl}^2 - x_{ik}x_{jk}x_{ll} \\
\Sigma[ij \mid klm] &= x_{ij}x_{kk}x_{ll}x_{mm} + x_{im}x_{jm}x_{kl}^2 - x_{im}x_{jl}x_{kl}x_{km} - x_{il}x_{jm}x_{kl}x_{km} + \\
&\quad x_{il}x_{jl}x_{km}^2 - x_{im}x_{jm}x_{kk}x_{ll} + x_{im}x_{jk}x_{km}x_{ll} + x_{ik}x_{jm}x_{km}x_{ll} - \\
&\quad x_{ij}x_{km}^2x_{ll} + x_{im}x_{jl}x_{kk}x_{lm} + x_{il}x_{jm}x_{kk}x_{lm} - x_{im}x_{jk}x_{kl}x_{lm} - \\
&\quad x_{ik}x_{jm}x_{kl}x_{lm} - x_{il}x_{jk}x_{km}x_{lm} - x_{ik}x_{jl}x_{km}x_{lm} + 2x_{ij}x_{kl}x_{km}x_{lm} + \\
&\quad x_{ik}x_{jk}x_{lm}^2 - x_{ij}x_{kk}x_{lm}^2 - x_{il}x_{jl}x_{kk}x_{mm} + x_{il}x_{jk}x_{kl}x_{mm} + \\
&\quad x_{ik}x_{jl}x_{kl}x_{mm} - x_{ij}x_{kl}^2x_{mm} - x_{ik}x_{jk}x_{ll}x_{mm} \\
&\ \vdots
\end{aligned}
$$

Gaussian CI models

Definition

The *model* of a set of CI constraints is the set of all PD matrices which satisfy them.

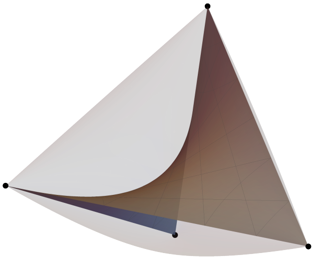


Figure: Model of $\Sigma[12|3] = a - bc = 0$ in the space of 3×3 correlation matrices.

Models and inference

Consider two sets of CI statements $\mathcal{P}$ and $\mathcal{Q}$:

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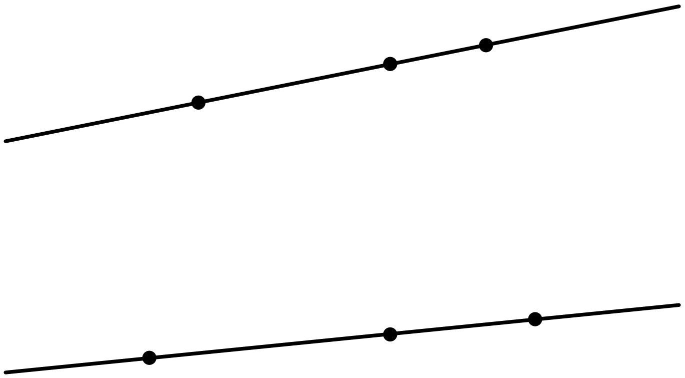
Reasoning about CI statements in normally distributed random variables is the same as reasoning about the vanishing of very special kinds of determinants on very special kinds of varieties inside the positive definite matrices.

For ancient geometers: conditional independence ≈ collinearity



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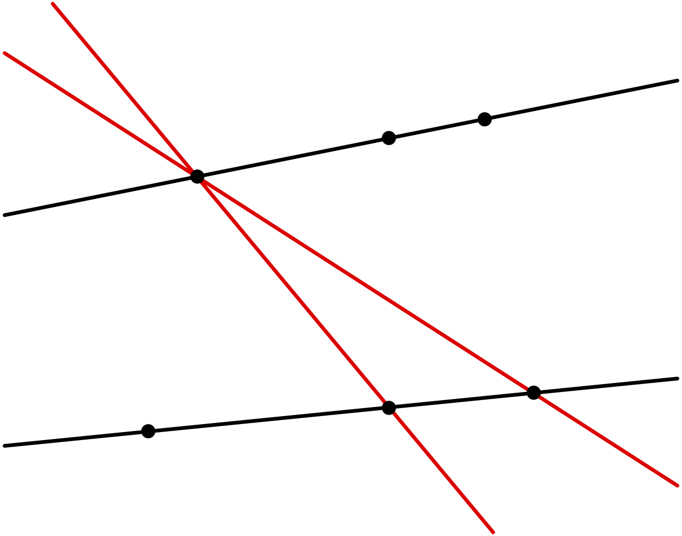

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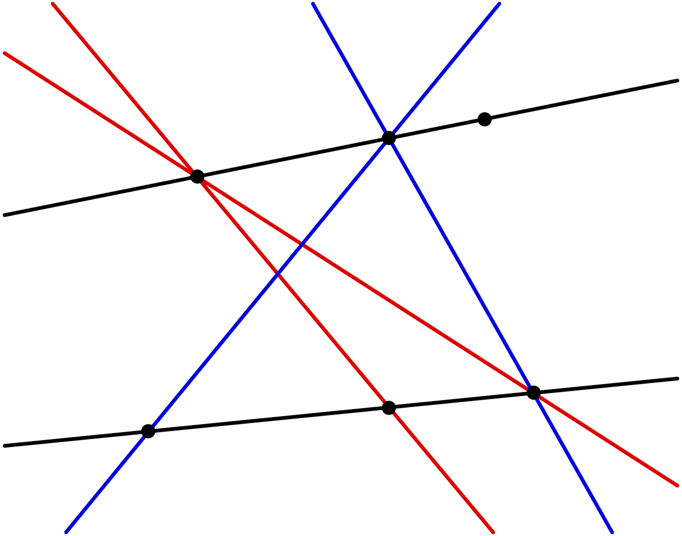

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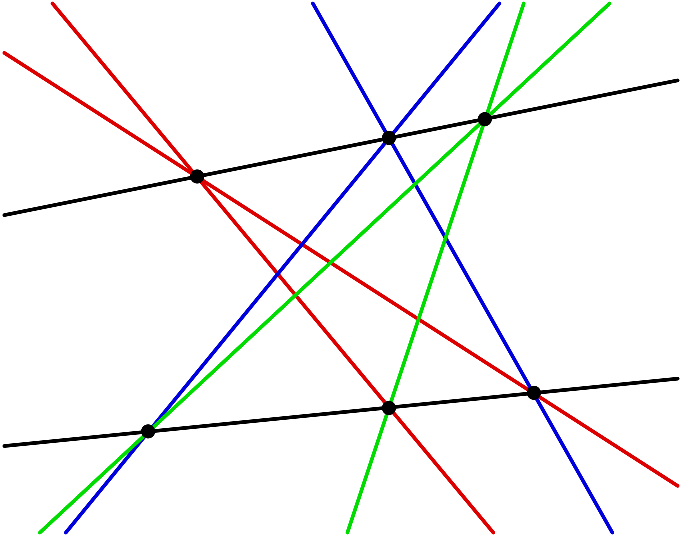

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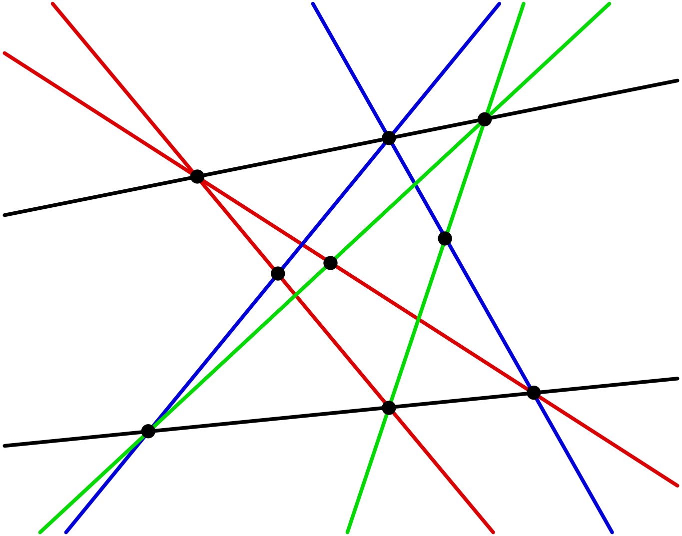

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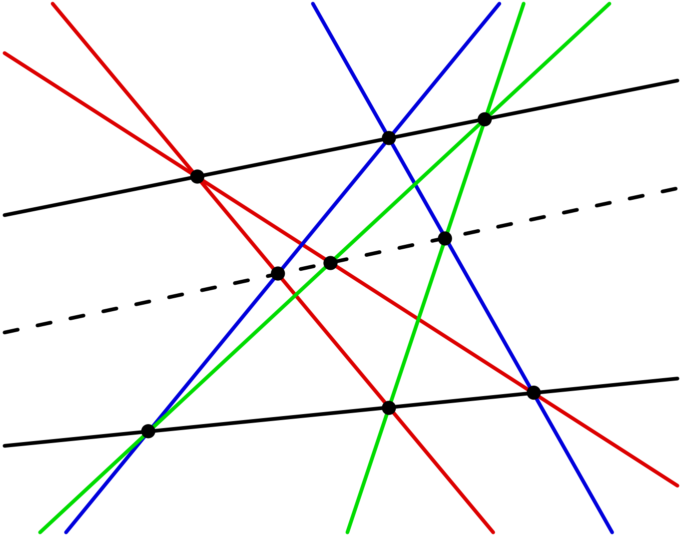

Normal form for proofs and refutations

Let $f_i \in \mathbb{Z}[t_1, \ldots, t_k]$ be integer polynomials in finitely many variables.

Theorem (Tarski's transfer principle)

If a polynomial system $\{f_i \bowtie_i 0\}$, where $\bowtie_i \in \{=, \neq, <, \leq, \geq, >\}$, has a solution over $\mathbb{R}$, then it has a solution in a finite real extension of $\mathbb{Q}$.

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→ If $\bigwedge \mathcal{P} \Rightarrow \bigvee \mathcal{Q}$ is false, there exists a counterexample matrix Σ with algebraic entries.

$[12|\,] \wedge [12|3] \Rightarrow [13|\,]$ is false and a counterexample is

$$\begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1 \end{pmatrix}.$$

Normal form for proofs and refutations

Let $f_i, g_j, h_k \in \mathbb{Z}[t_1, \ldots, t_k]$ be integer polynomials in finitely many variables.

Theorem (Positivstellensatz)

A polynomial system $\{f_i = 0, g_j \geq 0, h_k \neq 0\}$ *is infeasible if and only if there exist* $f \in \mathrm{ideal}(f_i)$, $g \in \mathrm{cone}(g_j)$ *and* $h \in \mathrm{monoid}(h_k)$ *such that* $g + h^2 = f$.

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→ If $\bigwedge \mathcal{P} \Rightarrow \bigvee \mathcal{Q}$ is true, there exists an algebraic proof for it with integer coefficients.

$[12||] \wedge [12|3] \Rightarrow [13||] \vee [23||]$ is true and a proof is the final polynomial

$$\Sigma[13||] \cdot \Sigma[23||] = \Sigma[3] \cdot \Sigma[12||] - \Sigma[12|3].$$

Computer algebra proves laws of probabilistic reasoning

The following inference rule is valid for all positive definite 5×5 matrices:

$$[12\,|\,]\wedge[14\,|\,5]\wedge[23\,|\,5]\wedge[35\,|\,1]\wedge[45\,|\,2]\wedge[15\,|\,23]\wedge[34\,|\,12]\wedge[24\,|\,135] \;\Rightarrow\; [25\,|\,]\vee[34\,|\,].$$

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$$\begin{gathered}
[25|\,][34|\,] \cdot [1][2][3][15] = \\
\big(cd^2egr + bd^2fgr - ad^2grh - 2cd^2e^2i - 2bd^2efi - 2pdfgri + 2ad^2ehi + 2pdefi^2 - 2pdqhi^2 + 2pcqi^3 + \\
2pdqrij - 2pbqi^2j - pcegrt + pbfgrt + pagrht + 2pce^2it - 2pcqrit + 2pbqhit - 2paehit\big) \cdot [12|\,] + \\
\big(pdqer + pbqgr - 2pbqei\big) \cdot [14|5] - \big(pcdqr + p^2fgr - 2pbcqi + 2pb^2qj - 2p^2qrj\big) \cdot [23|5] + \\
\big(cdqgr - 2cdqei + 2pqghi - 2pqfi^2 - pqgrj + 2pqeij - 2pe^2ft + 2pqfrt\big) \cdot [35|1] + \\
\big(pd^2er - 2pbdei + p^2gri + 2pb^2et - 2p^2ert\big) \cdot [45|2] - \big(2pdfi - 2pbft\big) \cdot [15|23] - \\
\big(d^2gr - 2d^2ei - pgrt + 2peit\big) \cdot [34|12] - 2pqi \cdot [24|135].
\end{gathered}$$

Computer algebra proves laws of probabilistic reasoning

```
R = QQ[p,a,b,c,d, q,e,f,g, r,h,i, s,j, t];
X = genericSymmetricMatrix(R,p,5);
I = ideal(
  det X_{0}^{1}, det X_{0,3}^{2,3}, det X_{0,4}^{3,4},
  det X_{1,4}^{2,4}, det X_{2,0}^{4,0}, det X_{3,1}^{4,1},
  det X_{0,1,2}^{4,1,2}, det X_{2,0,1}^{3,0,1},
  det X_{1,0,2,4}^{3,0,2,4}
);
U = g*h*p*q*r*(p*t-d^2); -- [25||][34||]·[1][2][3][15] ∈ monoid(𝒱)
U % I --> 0, meaning monoid(𝒱) ∩ ideal(𝒱) ≠ ∅ in ℚ[X]
-- Get a proof that U is in I:
G = gens I; -- the equations generating ideal(𝒱)
H = U // G; -- linear combinators for U from G
U == G*H --> true
```

Consistency checking is hard

The complexity class $\exists\mathbb{R}$ contains all decision problems which can be reduced in polynomial time to the feasibility of a semialgebraic set:

- polynomial optimization
- computational geometry
- algebraic statistics ...

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Theorem

The problem of deciding whether a general CI model is non-empty is complete for $\exists\mathbb{R}$.

Consistency certification is hard

Šimeček's Question (2006)

Does every non-empty Gaussian CI model contain a rational point?

Or: can every wrong inference rule be refuted over $\mathbb{Q}$?

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Model M85

$$\begin{matrix} 1 & a & b & c \\ a & 1 & d & e \\ b & d & 1 & f \\ c & e & f & 1 \end{matrix}$$

Where:

$$a = \frac{3}{632836}\sqrt{1107463},$$

$$b = 10c = \frac{100}{158209}\sqrt{1107463}$$

$$d = 10e = \frac{3}{4}, f = \frac{1}{10}$$

$$\begin{pmatrix} 1 & -1/17 & -49/51 & -7/17 \\ -1/17 & 1 & 1/3 & 1/7 \\ -49/51 & 1/3 & 1 & 3/7 \\ -7/17 & 1/7 & 3/7 & 1 \end{pmatrix}$$

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Theorem

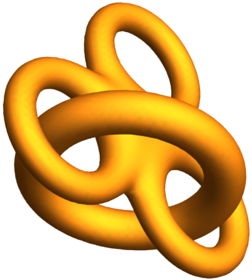
For every finite real extension $\mathbb{K}$ of $\mathbb{Q}$ there exists a CI model $\mathcal{M}$ such that $\mathcal{M} \neq \varnothing$ but $\mathcal{M} \cap \mathrm{PD}_N(\mathbb{K}) = \varnothing$.

Model topology can be bad

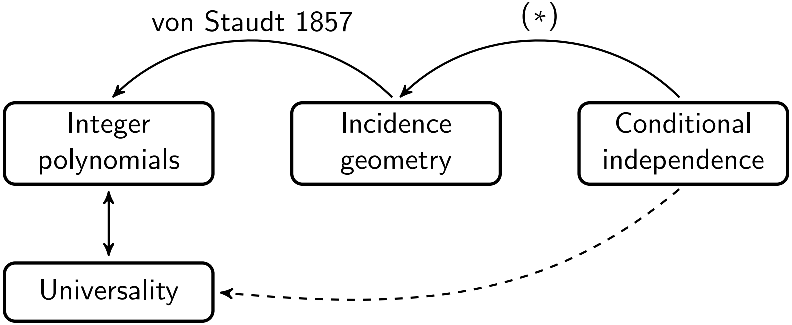
An oriented CI model is specified by sign constraints on partial correlations.

Theorem

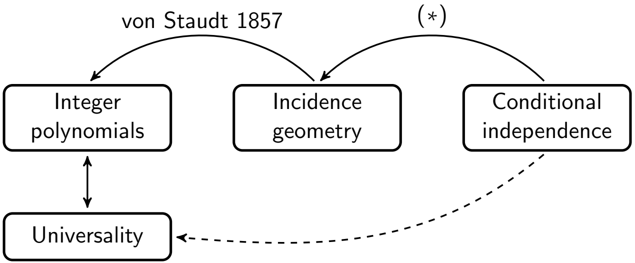
For every primary basic semialgebraic set Z there exists an oriented CI model $\mathcal{M}$ which is homotopy-equivalent to Z.



Universality theorems



Universality theorems



- Realization spaces of rank-3 matroids
- Realization spaces of 4-polytopes
- Nash equilibria of 3-person games
- Gaussian CI models with conditioning sets of size up to 3 ...

References

- Tobias Boege. "The Gaussian conditional independence inference problem". PhD thesis. OvGU Magdeburg, 2022.
- Jürgen Bokowski and Bernd Sturmfels. *Computational synthetic geometry*. Vol. 1355. Lecture Notes in Mathematics. Springer, 1989.
- Jürgen Richter-Gebert. *Perspectives on projective geometry. A guided tour through real and complex geometry*. Springer, 2011, pp. xxii + 571.
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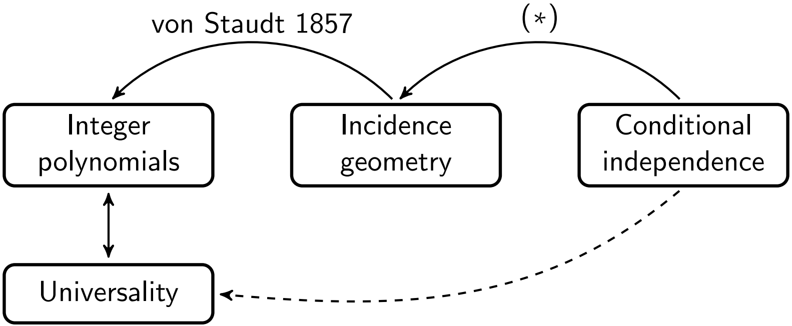
Universality theorems: Background



Theorem

To every polynomial system $\{f_i \bowtie 0\}$ there is a set of CI constraints which has a model over a field $\mathbb{K}/\mathbb{Q}$ if and only if the polynomial system has a solution in $\mathbb{K}$.

Very special polynomials

$\Sigma[ij\,|\,] = x_{ij} \to$ impose $x_{kl} = x_{km} = x_{lm} = 0$ on a correlation matrix, then:

$$
\begin{aligned}
\Sigma[ij\,|\,klm] = {} & x_{ij}x_{kk}x_{ll}x_{mm} + x_{im}x_{jm}\underline{x_{kl}^2} - x_{im}x_{jl}\underline{x_{kl}x_{km}} - x_{il}x_{jm}\underline{x_{kl}x_{km}} + x_{il}x_{jl}\underline{x_{km}^2} \\
& - x_{im}x_{jm}x_{kk}x_{ll} + x_{im}x_{jk}\underline{x_{km}}x_{ll} + x_{ik}x_{jm}\underline{x_{km}}x_{ll} - x_{ij}\underline{x_{km}^2}x_{ll} \\
& + x_{im}x_{jl}x_{kk}\underline{x_{lm}} + x_{il}x_{jm}x_{kk}\underline{x_{lm}} - x_{im}x_{jk}\underline{x_{kl}x_{lm}} - x_{ik}x_{jm}\underline{x_{kl}x_{lm}} \\
& - x_{il}x_{jk}\underline{x_{km}x_{lm}} - x_{ik}x_{jl}\underline{x_{km}x_{lm}} + 2x_{ij}\underline{x_{kl}x_{km}x_{lm}} + x_{ik}x_{jk}\underline{x_{lm}^2} \\
& - x_{ij}x_{kk}\underline{x_{lm}^2} - x_{il}x_{jl}x_{kk}x_{mm} + x_{il}x_{jk}\underline{x_{kl}}x_{mm} + x_{ik}x_{jl}\underline{x_{kl}}x_{mm} \\
& - x_{ij}\underline{x_{kl}^2}x_{mm} - x_{ik}x_{jk}x_{ll}x_{mm} \\
= {} & x_{ij} - \sum_{k=k,l,m} x_{ik}x_{jk} \quad = \quad x_{ij} - \left\langle \begin{pmatrix} x_{ik} \\ x_{il} \\ x_{im} \end{pmatrix}, \begin{pmatrix} x_{jk} \\ x_{jl} \\ x_{jm} \end{pmatrix} \right\rangle .
\end{aligned}
$$

The rest is 19th century projective geometry. Keyword: *von Staudt constructions*.

Covariance matrix simulating a projective plane

$$
\begin{array}{c} \\ \mathsf{p}_1 \\ \vdots \\ \mathsf{p}_n \\ \mathsf{l}_1 \\ \vdots \\ \mathsf{l}_m \\ \mathsf{x} \\ \mathsf{y} \\ \mathsf{z} \end{array}
\begin{array}{c}
\begin{array}{ccc|ccc|ccc} \mathsf{p}_1 & \cdots & \mathsf{p}_n & \mathsf{l}_1 & \cdots & \mathsf{l}_m & \mathsf{x} & \mathsf{y} & \mathsf{z} \end{array} \\
\left(\begin{array}{ccc|ccc|ccc}
p_1^* & & \langle p, p' \rangle & & & & p_1^x & p_1^y & p_1^z \\
& \ddots & & & \langle p, \ell \rangle & & & \vdots & \\
\langle p', p \rangle & & p_n^* & & & & p_n^x & p_n^y & p_n^z \\
\hline
& & & \ell_1^* & & \langle \ell, \ell' \rangle & \ell_1^x & \ell_1^y & \ell_1^z \\
& \langle \ell, p \rangle & & & \ddots & & & \vdots & \\
& & & \langle \ell', \ell \rangle & & \ell_m^* & \ell_m^x & \ell_m^y & \ell_m^z \\
\hline
p_1^x & & p_n^x & \ell_1^x & & \ell_m^x & x^* & 0 & 0 \\
p_1^y & \cdots & p_n^y & \ell_1^y & \cdots & \ell_m^y & 0 & y^* & 0 \\
p_1^z & & p_n^z & \ell_1^z & & \ell_m^z & 0 & 0 & z^*
\end{array}\right)
\end{array}
$$