The Copy lemma and rank inequalities

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Combinatorial Coworkspace
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- Submodularity in general already follows from local submodularity:

$$f(i : j \mid K) := f(\{i\} \cup K) + f(\{j\} \cup K) - f(\{i, j\} \cup K) - f(K) \geq 0$$

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  for all $i, j \in N$, $K \subseteq N$.
- Think of the symbols (K) and $(i : j \mid K)$ as dual vectors in $(\mathbb{R}^{2^N})^*$.

Some interesting classes of submodular functions

- **Dimensions of subspace arrangements**: D a skew field, V a finite-dimensional (left) vector space over D and $(U_i : i \in N)$ a collection of subspaces:

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- **Logarithmic principal minors**: Σ an $n \times n$ positive definite matrix:

$$K \mapsto \log \det \Sigma_{K,K}.$$

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- **Shannon entropies**: $(X_i : i \in N)$ jointly distributed discrete random variables with X_i taking values in Q_i:

$$K \mapsto H(X_K) \coloneqq - \sum_{q_k \in Q_k : k \in K} \Pr[X_k = q_k : k \in K] \cdot \log \Pr[X_k = q_k : k \in K].$$

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What else?

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- Inequalities on the submodular part of the "tropical affine flag variety". [ARVY25]

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It is enough to "homogenize" and close up in the euclidean topology to get the convex conic closure of $\mathcal{S}_N$.

Elementary rank inequalities

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$$[G : H_1][G : H_1 \cap H_3][G : H_2 \cap H_3]^2 \geq [G : H_3][G : H_1 \cap H_2 \cap H_3]^2.$$

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- The reason is obvious:

$$(1)-(3)+(1,3)+2(2,3)-2(1,2,3) = (1:2)+(1:3 \mid 2)+(1:2 \mid 3) \in \mathrm{Submod}_3^*.$$

Some non-elementary rank inequalities

- The Zhang–Yeung inequality [ZY98] also holds for **all** the example classes:

$$(1:2) + 2(3:4 \mid 1) + (3:4 \mid 2) + (1:3 \mid 4) + (1:4 \mid 3) \geq (3:4).$$

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- In fact, for every $s \geq 0$ the following generalization holds [Mat07]:

$$s\left[\mathbf{(1:2)} + \mathbf{(3:4 \mid 1)} + \mathbf{(3:4 \mid 2)} - \mathbf{(3:4)}\right] +$$
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- The highlighted part is the Ingleton functional which is always non-negative for multilinear functions but can be negative for any of the other example classes!

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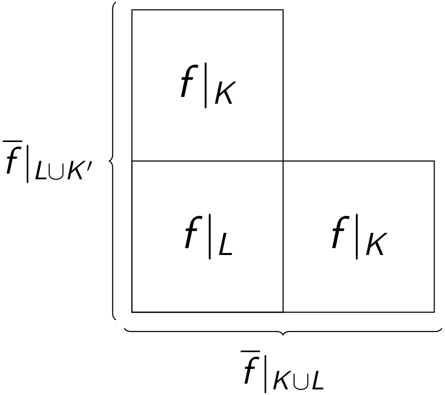

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Copy lemma template

Fix a family $(C_N : N \text{ finite})$, $C_N \subseteq \mathrm{Submod}_N$, of cones of interest. Suppose that $\pi_L(C_N) \subseteq C_L$ for all $L \subseteq N$. If $f \in C_{K \cup L}$ then there exists a selfadhesive extension $\overline{f} \in \mathrm{Adhe}_N(f; L) \cap C_{K \cup L \cup K'}$.

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- For Shannon entropies use the conditional product [Stu21].
- For PD matrices this works [Boe23]:

$$\Sigma = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix} \mapsto \begin{pmatrix} A & B & B \\ B^T & D & \Lambda \\ B^T & \Lambda^T & D \end{pmatrix}, \quad \text{with } \Lambda = B^T A^{-1} B.$$

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- We have $N = \{1,2,3,4\}$. Set $K = \{1,2\}$, $L = \{3,4\}$, and $K' = \{1',2'\}$.
- This function is evidently non-negative on $\mathrm{Submod}_{\{1,2,3,4,1',2'\}}$:

$$(1:2 \mid 1') + (1:1' \mid 3) + (1:1' \mid 4) + (1:1' \mid 234) + 3\cdot(1:2' \mid 1'34) + \\ (1':2 \mid 3) + (1':2 \mid 4) + (1':2 \mid 134) + 3\cdot(2:2' \mid 11'34) + \\ (3:4 \mid 11') + (3:4 \mid 1'2) + (1':3 \mid 124) + (1':4 \mid 12).$$

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$$(3:4 \mid 11') + (3:4 \mid 1'2) + (1':3 \mid 124) + (1':4 \mid 12).$$

- It is equal to the claimed expression modulo the Copy relations:

$$(34) + (11'22'34) = (1234) + (1'2'34),$$
$$(1') = (1),\ \ (13) = (1'3),\ \ (14) = (1'4),\ \ (1'34) = (134).$$

Higher selfadhesivity

The Copy lemma can "amplify" the submodular inequalities.

Lemma

The function $f \in C_{K \cup L}$ has a selfadhesive extension in $C_{K \cup L \cup K'}$ with respect to L if and only if $f \in \pi_{K \cup L}(\mathrm{Adhe}_N(L) \cap C_{K \cup L \cup K'})$ where $\mathrm{Adhe}_N(L)$ is the linear space given by

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- The selfadhesion of C_N is $C_N^{sa} := \bigcap_{L \subseteq N} \pi_{K \cup L}(\mathrm{Adhe}_N(L) \cap C_{K \cup L \cup K'})$.

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- So Copy lemma $\implies$ $C_N \subseteq \mathrm{Submod}_N^{\mathrm{sa}}$.

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- So Copy lemma $\implies$ $C_N \subseteq \mathrm{Submod}_N^{\mathrm{sa}}$.
- The Vámos matroid is in $\mathrm{Submod}_8 \setminus \mathrm{Submod}_8^{\mathrm{sa}}$ because it violates Zhang–Yeung.

Higher selfadhesivity

- We can iterate this: let

$$\mathrm{Submod}_N^{0\cdot\mathrm{sa}} := \mathrm{Submod}_N,$$
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- Higher selfadhesions of Submod_N form a descending chain of polyhedral cones.
- Very little is known about the limit object

$$C_N \subseteq \mathrm{Submod}_N^{\omega\cdot \mathrm{sa}} := \bigcap_{k<\omega} \mathrm{Submod}_N^{k\cdot \mathrm{sa}},$$

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- We can iterate this: let

$$\mathrm{Submod}_N^{0\cdot\mathrm{sa}} := \mathrm{Submod}_N,$$
$$\mathrm{Submod}_N^{(k+1)\cdot\mathrm{sa}} := (\mathrm{Submod}_N^{k\cdot\mathrm{sa}})^{\mathrm{sa}}.$$

- Higher selfadhesions of Submod_N form a descending chain of polyhedral cones.
- Very little is known about the limit object

$$C_N \subseteq \mathrm{Submod}_N^{\omega\cdot\mathrm{sa}} := \bigcap_{k<\omega} \mathrm{Submod}_N^{k\cdot\mathrm{sa}},$$

except that it is *non-polyhedral*. [Mat07]

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