The Ingleton inequality for random variables

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Combinatorial Coworkspace
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Matroids

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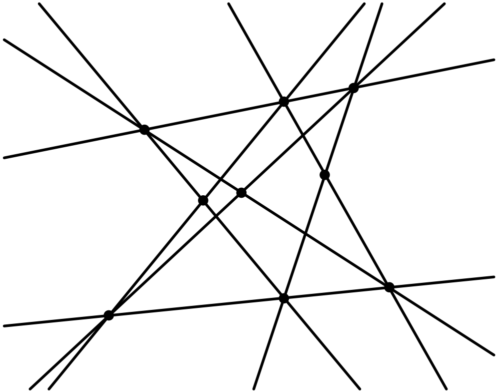

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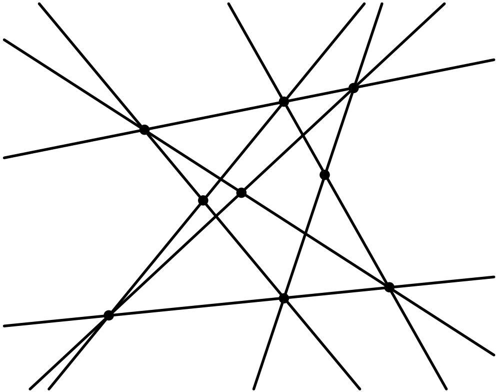

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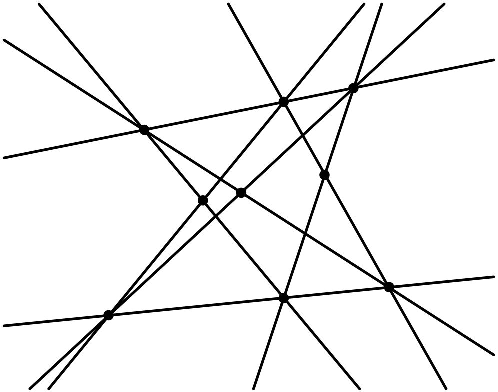
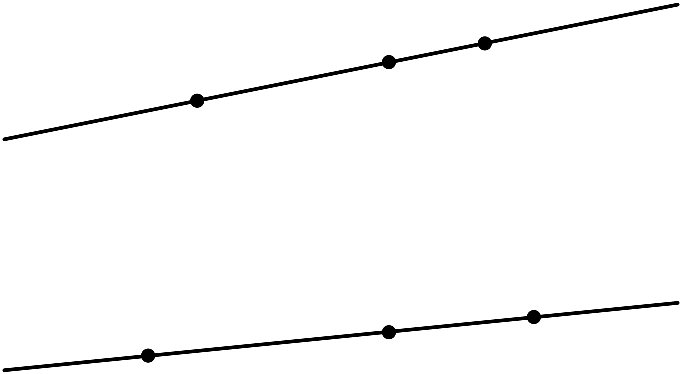
  - For example the matroid of a set of points in the projective plane records which triples of points lie on a line.
- Non-realizability of matroids captures the (non-obvious) laws of projective geometry.



Laws of geometry



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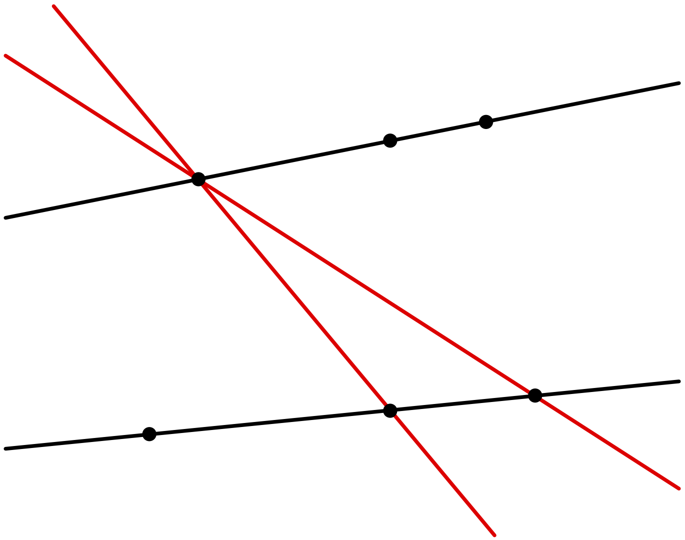

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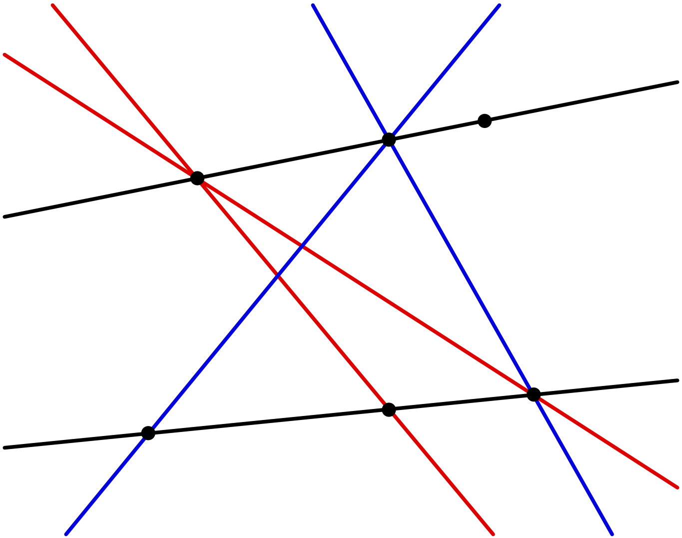

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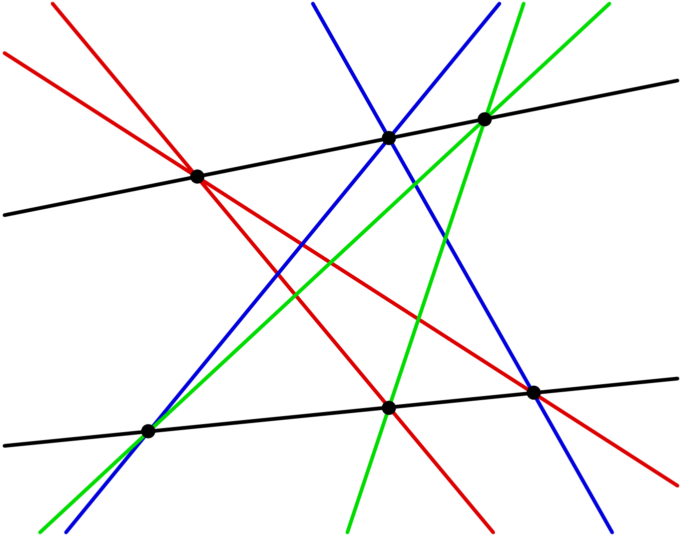

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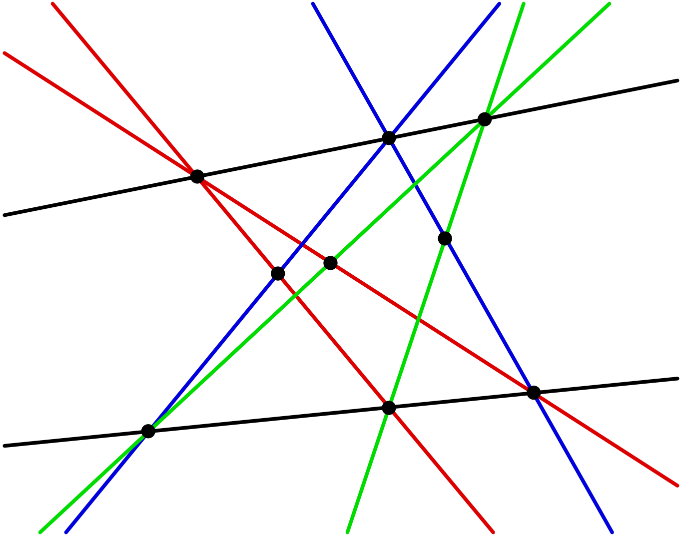

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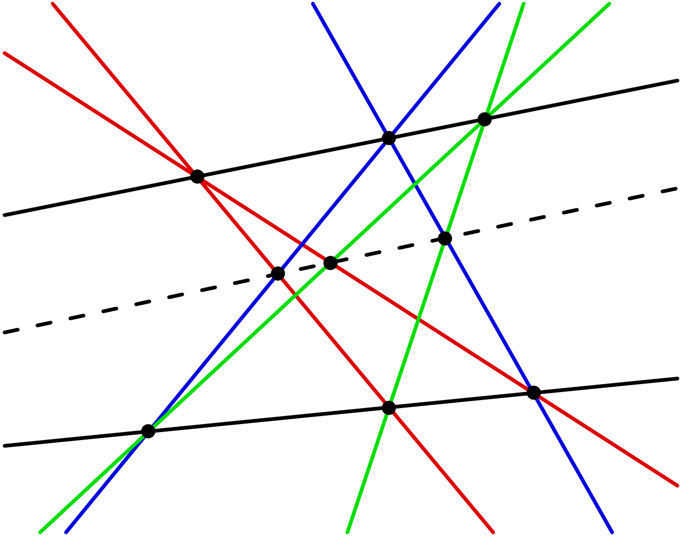

Entropy

Let X be a random variable taking finitely many values $\{1, \ldots, d\}$ with positive probabilities. Its *Shannon entropy* is

$$H(X) := \sum_{i=1}^{d} p(X = i) \log {}^{1}\!/_{p(X = i)}.$$

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- H is continuous on $\Delta(d)$ and analytic on the interior.
- A random vector $X \in \Delta(d_1, \ldots, d_n)$ is a random variable in $\Delta(\prod_{i=1}^{n} d_i)$, so the definition of H extends to vectors.
- For a random vector $X = (X_1, \ldots, X_n)$ we have 2^n marginals and we collect their entropies in an entropy profile $h_X : 2^{[n]} \to \mathbb{R}$.
  - For example (X, Y) has entropy profile $(0, H(X), H(Y), H(X, Y)) \in \mathbb{R}^4$.

Entropy as information

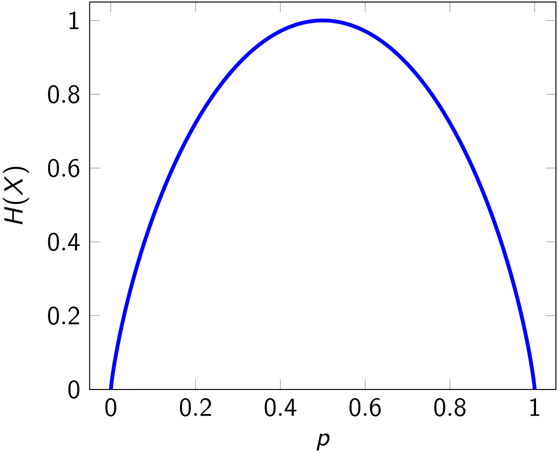

Figure: Entropy of a binary random variable X as a function of $p = p(X = \text{heads})$.

Independence: geometry $\leftrightarrow$ information theory

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Rank condition	Matroid concept	Information theory concept
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Even though entropy is a transcendental function, many of these conditions are **polynomial** in the probabilities $\rightarrow$ algebraic statistics.

The entropy region and information inequalities

Let $\mathbf{H}_n^* \subseteq \mathbb{R}^{2^n}$ consist of all h_X where X is an n-variate discrete random vector. $\mathbf{H}_n^*$ is the image of $\bigcup_{d_1=1}^{\infty} \cdots \bigcup_{d_n=1}^{\infty} \Delta(d_1, \ldots, d_n)$ under the transcendental map $X \mapsto h_X$.

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Theorem ([Mat07])

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- Elements of the dual cone (linear information inequalities) can give bounds for optimization problems.

Ingleton inequality

Let A, B, C, D be subspaces in a finite-dimensional vector space.
Then the Ingleton inequality holds for $h = \dim$:

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These are conditional linear information inequalities and they can tell apart honest boundary parts of $\mathbf{H}_n^*$ from fake boundary parts on $\overline{\mathbf{H}_n^*}$.

Conditional Ingleton inequalities

Theorem ([KR13] & [Stu21] & [Boe23])

Up to symmetry there are precisely ten minimal sets of conditional independence assumptions on four random variables which ensure $I \geq 0$.

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Which of these laws holds on $\overline{\mathbf{H}_N^*}$? (Some do, some don't . . .)

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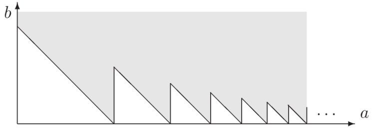

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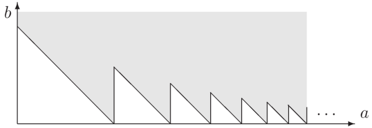

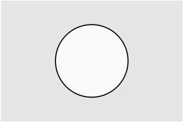

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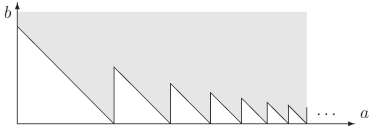

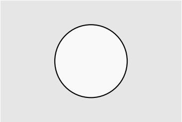

References

[Boe23] Tobias Boege. “No Eleventh Conditional Ingleton Inequality”. In: *Experimental Mathematics* (2023). DOI: 10.1080/10586458.2023.2294827.

[KR13] Tarik Kaced and Andrei Romashchenko. “Conditional information inequalities for entropic and almost entropic points”. In: *IEEE Trans. Inf. Theory* 59.11 (2013), pp. 7149–7167. ISSN: 0018-9448. DOI: 10.1109/TIT.2013.2274614.

[Mat06] Frantisek Matús. “Piecewise linear conditional information inequality”. In: *IEEE Trans. Inf. Theory* 52.1 (2006), pp. 236–238. ISSN: 0018-9448. DOI: 10.1109/TIT.2005.860438.

[Mat07] František Matúš. “Two constructions on limits of entropy functions.”. In: *IEEE Trans. Inf. Theory* 53.1 (2007), pp. 320–330. ISSN: 0018-9448. DOI: 10.1109/TIT.2006.887090.

[Stu21] Milan Studený. “Conditional independence structures over four discrete random variables revisited: conditional ingleton inequalities”. In: *IEEE Trans. Inf. Theory* 67.11 (2021), pp. 7030–7049. ISSN: 0018-9448. DOI: 10.1109/TIT.2021.3104250.