Colored Gaussian DAG models

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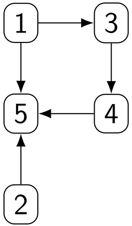
Gaussian DAG models

- A linear structural equation model defines random variables X recursively via a directed acyclic graph $G = (V, E)$ and Gaussian noise:

$$X_j = \sum_{i \in \mathrm{pa}(j)} \lambda_{ij} X_i + \varepsilon_j, \quad \varepsilon_j \sim \mathcal{N}(0, \omega_j).$$



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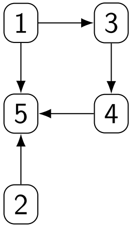

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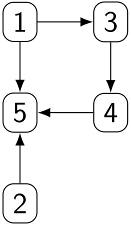

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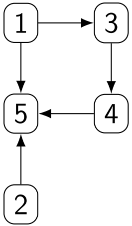

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- All such matrices form the model $\mathcal{M}(G) \subseteq \mathrm{PD}_V$.

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- Model equivalence $\mathcal{M}(G) = \mathcal{M}(H)$ is combinatorially characterized: if and only if G and H have the same skeleton and v-structures.
  - Markov equivalence = ambiguity about the direction of causality.

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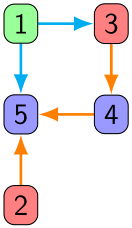

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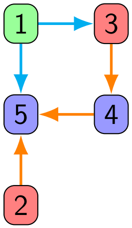

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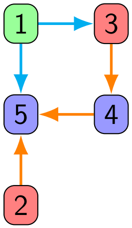

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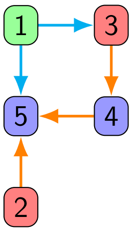
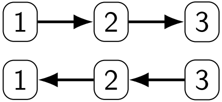
- Vertex-only colorings correspond to partial homoscedasticity [WD23].



Coloring can disambiguate the causal structure

- Coloring reduces Markov-equivalence classes which eases causal discovery.



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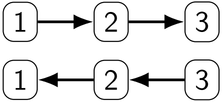

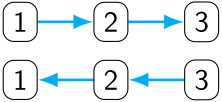

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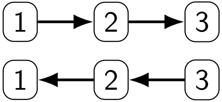

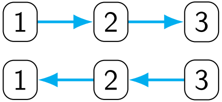

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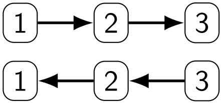

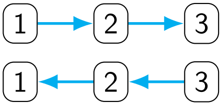

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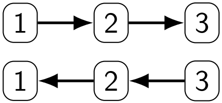

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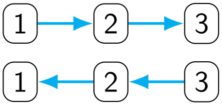

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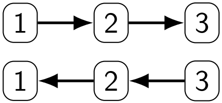

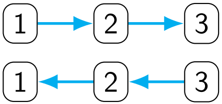

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- Not invariant anymore.

Parameter identifiability revisited

- It follows from the recursive factorization and some linear algebra that

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- A set A is identifying for a vertex j resp. edge ij if

$$\omega_j = \omega_{j|A}(\Sigma) \text{ resp. } \lambda_{ij} = \lambda_{ij|A}(\Sigma)$$

for all $\Sigma \in \mathcal{M}(G)$.

Parameter identifiability revisited

Theorem

Let $G = (V, E)$ be a DAG. Then:

- $\omega_j = \omega_{j|A}(\Sigma)$ *for every* $\Sigma \in \mathcal{M}(G)$ *if and only if* $\mathrm{pa}(j) \subseteq A \subseteq V \setminus \overline{\mathrm{de}}(j)$. [WD23]

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- *If* $ij \notin E$, *then* $\lambda_{ij} = 0 = \lambda_{ij|A}(\Sigma)$ *for every* $\Sigma \in \mathcal{M}(G)$ *if and only if* $A \setminus i$ *d-separates* i *and* j *in* G. *[Folklore]*

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- The polynomials $\mathrm{vcr}(i|A, j|B) = |\Sigma_A||\Sigma_B|(\omega_{i|A} - \omega_{j|B})$ resp. $\mathrm{ecr}(ij|A, kl|B) = |\Sigma_A||\Sigma_B|(\lambda_{ij|A} - \lambda_{kl|B})$ vanish on the model $\mathcal{M}(G, c)$ whenever $c(i) = c(j)$ resp. $c(ij) = c(kl)$ and A and B are identifying.

Model geometry

Theorem

For every colored DAG (G, c) *the model* $\mathcal{M}(G, c)$ *has irreducible Zariski closure and is a smooth submanifold of* PD_V*. It is diffeomorphic to an open ball of dimension* $\mathrm{vc} + \mathrm{ec}$ *(the number of vertex- and edge-color classes).*

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- $I_c = \langle \mathrm{vcr}(i|\mathrm{pa}(i), j|\mathrm{pa}(j)) : c(i) = c(j)\rangle + \langle \mathrm{ecr}(ij|\mathrm{pa}(j), kl|\mathrm{pa}(l)) : c(ij) = c(kl)\rangle$ *is the coloring ideal of* G*,*
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- Resolves the colored generalization of a conjecture of Sullivant; see also [RP14].

Implicitization up to saturation

Lemma

Let R, R' be rings, $S \subseteq R$ multiplicatively closed, and:

- *maps $\phi : R \to R'$ and $\psi : R' \to S^{-1}R$ with $\psi \circ \phi = \mathrm{id}_R$,*
- *for a prime ideal $I' = \langle f_1, \dots, f_k \rangle$, write $\psi(f_i) = {}^{g_i}/_{h_i}$ and set $J = \langle g_i \rangle$.*

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- Knowing a parametrization and generators for the vanishing ideal up to saturation is sufficient in practice for model distinguishability.
- Conceivable to extend to non-linear equations and inequalities.

Faithfulness

Fix a colored DAG (G, c) and $\Sigma \in \mathcal{M}(G, c)$.

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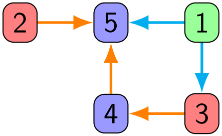
- The example on the right colors vertices and edges.
  The generic matrix in the model satisfies $1 \perp\!\!\!\perp 4 \mid 5$.
  No faithful distribution!



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Theorem ([WD23])

If (G, c) and (H, c) are vertex-colored DAGs, then $\mathcal{M}(G, c) = \mathcal{M}(H, c)$ if and only if G and H are Markov-equivalent and $\mathrm{pa}_G(j) = \mathrm{pa}_H(j)$ for all $j \in V$ with $|c^{-1}(j)| \geq 2$.

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Theorem

If (G, c) and (H, d) are two BPEC-DAGs, then $\mathcal{M}(G, c) = \mathcal{M}(H, d)$ implies $(G, c) = (H, d)$. In particular, the Markov-equivalence classes of BPEC-DAGs are singletons and the causal structure is identifiable.

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