Algebraic matroids and group configurations

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Outline

Recognizing linear matroids

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Universality theorem (MacLane (1936)?)

The recognition problem is just as hard as deciding if a system of Diophantine equations has a solution over F.

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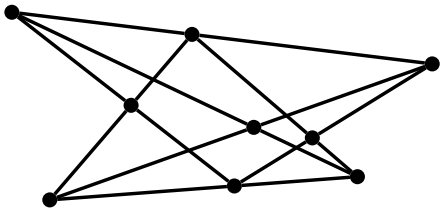

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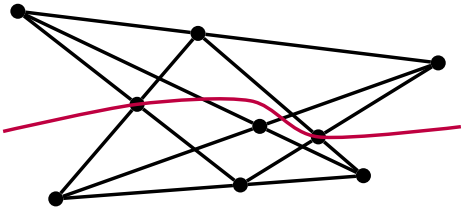

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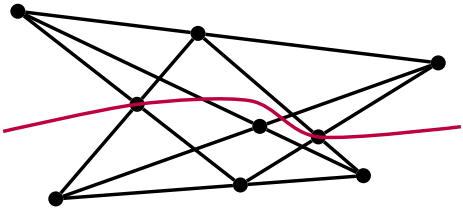
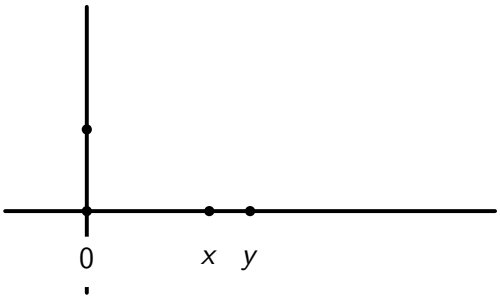


von Staudt (1857)

The existential theory of F reduces to constructing point-line configurations.

Von Staudt constructions



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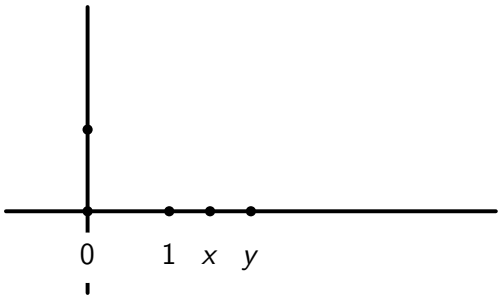

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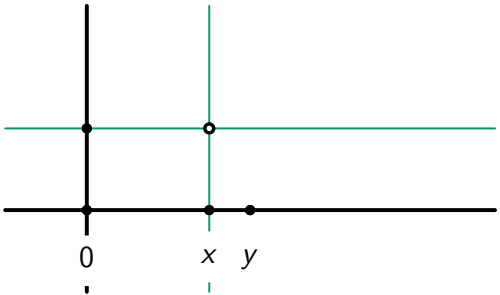

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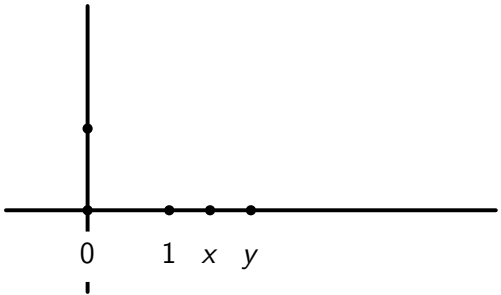

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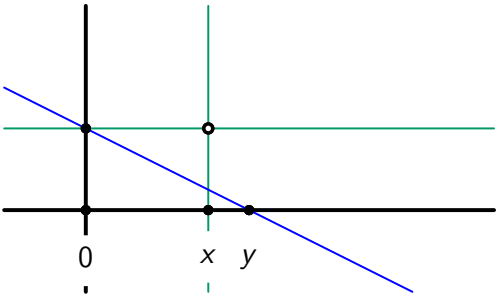

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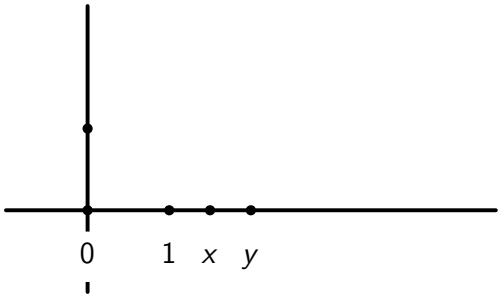

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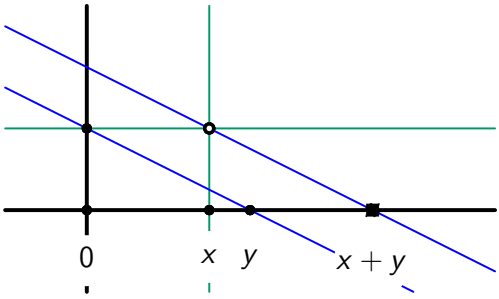

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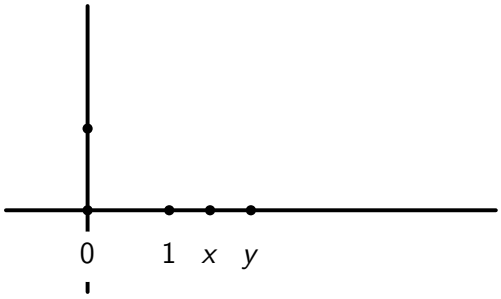

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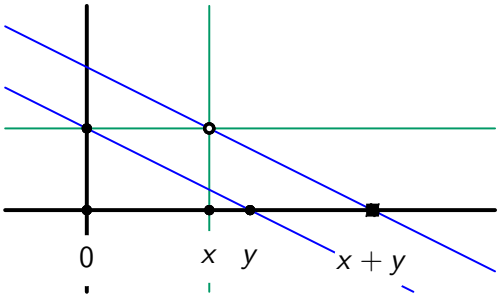

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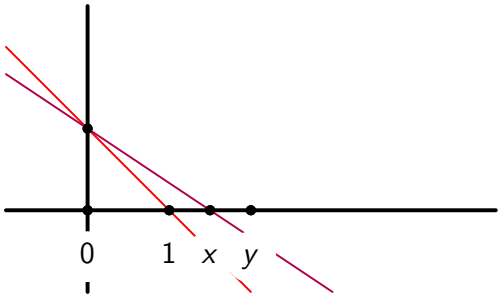

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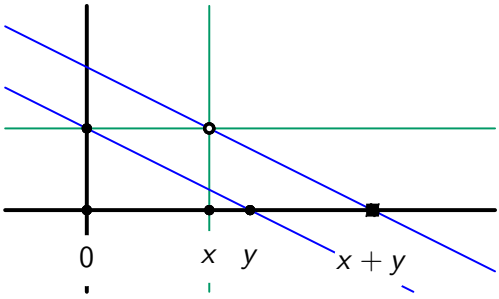

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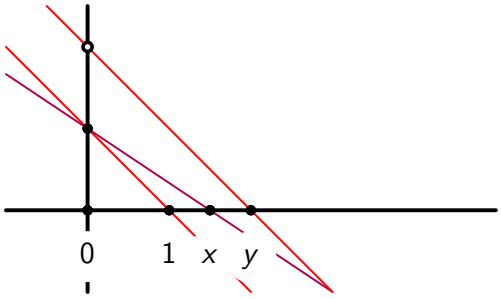

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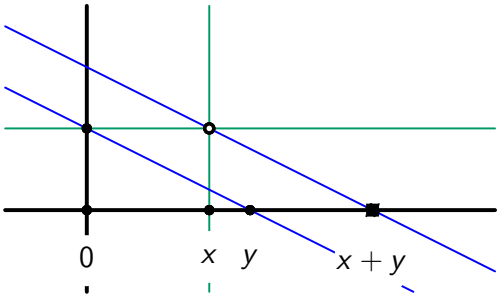

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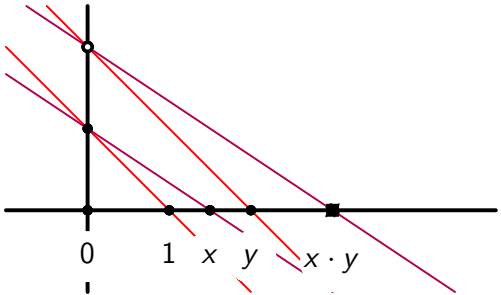

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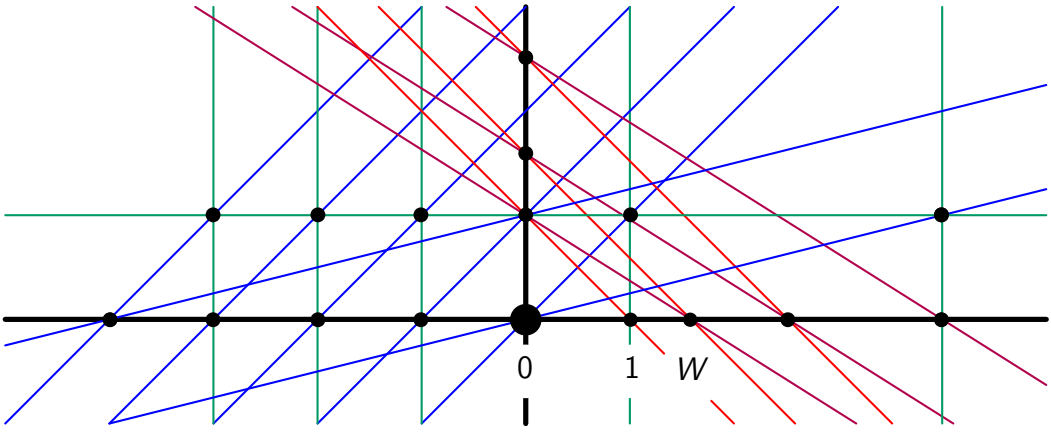
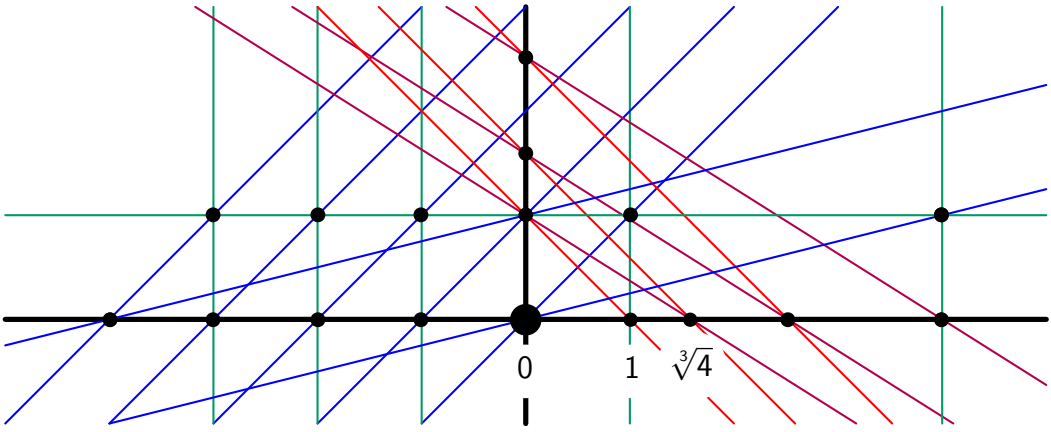
Where is Waldo?



Where is Waldo? On the cube root of 4!



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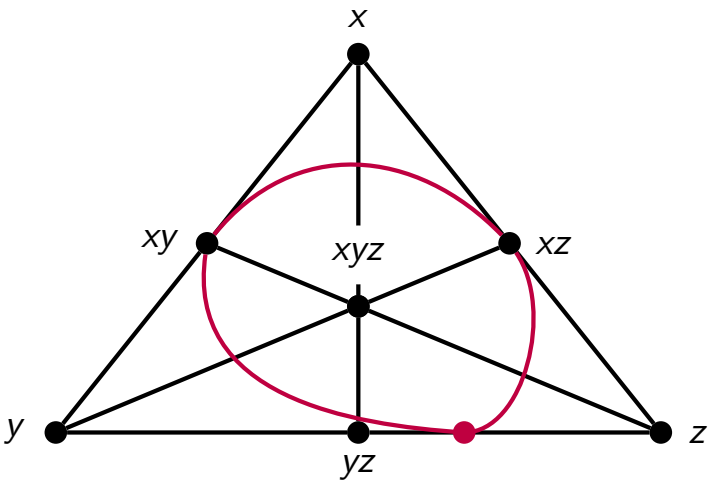
Non-Fano is algebraic over $\mathbb{F}_2$



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Lindström (1985)

For each prime $p > 0$ there is a matroid M_p which is algebraic over a field F if and only if $\mathrm{char}\, F = p$.

☞ Can reduce AlgMat_p to AlgMat by transforming M to $M \oplus M_p$.

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- Unclear how to perform the search or conclude that there is no representation.
- Also unclear how to encode the execution of a universal Turing machine in this.

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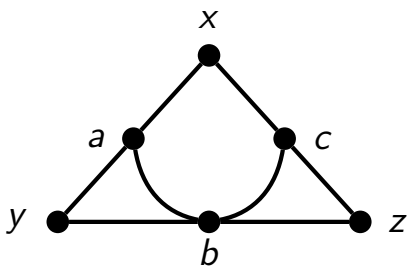

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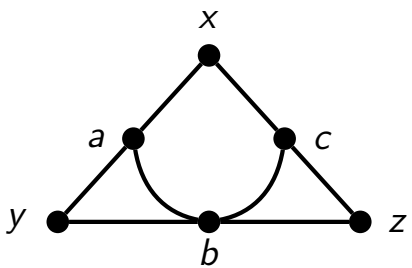

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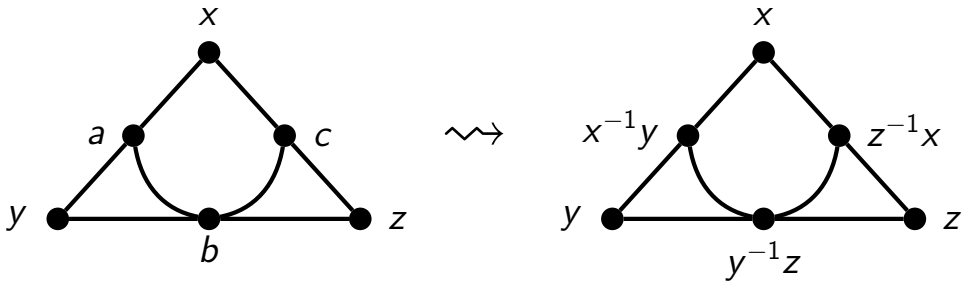

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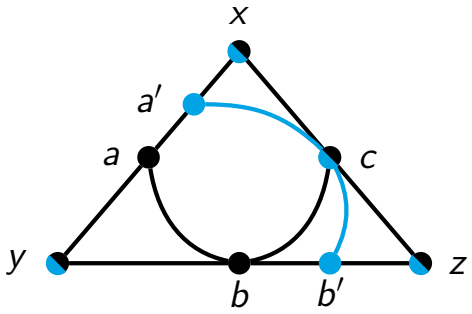
Evans & Hrushovski (1991)

Consider the augmented (double) group configuration below. There exist non-zero $\beta, \gamma \in \mathrm{End}_F(G)$ such that $\mathrm{acl}(b') = \mathrm{acl}(\beta(y) * \gamma(z))$.



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- $L_F(\mathbb{G}_{\mathsf{a}})$ is the skew field $F(\sigma)$ where $\sigma\colon F \to F$ is the Frobenius,
- $L_F(\mathbb{G}_{\mathsf{m}}) = \mathbb{Q}$,
- $L_F(\mathbb{E}) \in \left\{ \mathbb{Q}(\sqrt{-d}),\ \left(\frac{a,b}{\mathbb{Q}}\right) \right\}$.

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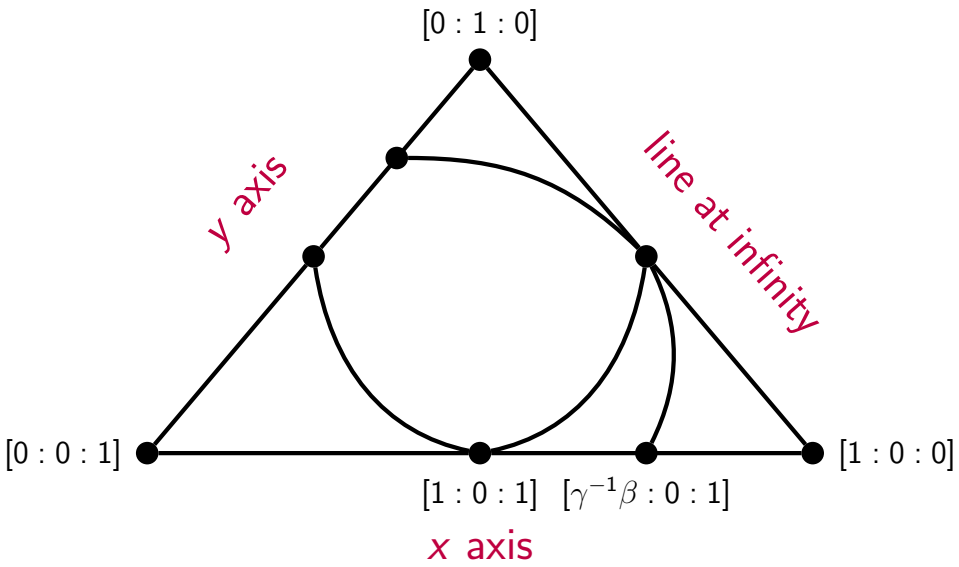
- The group configuration acts as a projective basis.

Group configuration as a projective basis



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- Then $w \in \mathbb{Q} \iff xw = wx \wedge yw = wy$ is an existential definition of $\mathbb{Q}$.

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- Instead, there is a group of definable automorphisms through which the coordinate skew fields of $\mathbb{G}_{\mathrm{a}}$ and $\mathbb{G}_{\mathrm{m}}$ can interact.

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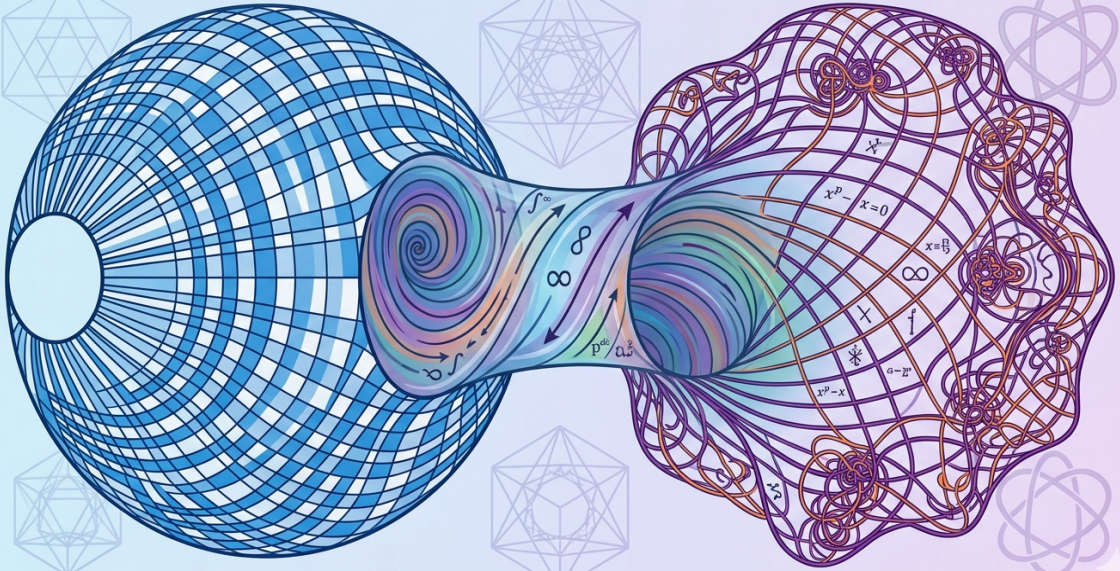
Pheidas (1991) & Videla (1994)

The existential theory of the field $\mathbb{F}_p(t)$ with named generator t is undecidable.



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