Real birational implicitization for statistical models

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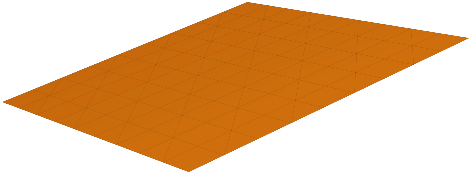
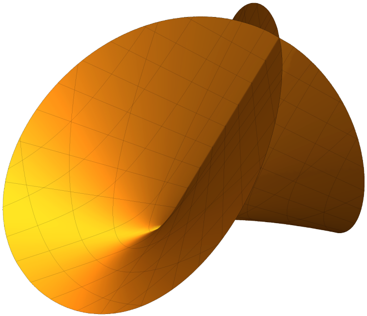
  - . . . for structure learning

The main idea





$$\Theta \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathcal{M}$$

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If the geometry of Θ is simple, then so is that of $\mathcal{M}$.

Geometric birational implicitization

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If $\Theta = \{\check{x} \in \mathbb{R}^n : \check{f}_i(\check{x}) = 0,\ \check{p}_j(\check{x}) \geq 0,\ \check{u}_k(\check{x}) \neq 0,\ \check{s}_\ell(\check{x}) > 0\}$ *is non-empty,*

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$$\mathcal{M} = \alpha(\Theta) = \{x \in \mathbb{R}^n : f_i(x) = 0,\ p_j(x) \geq 0,\ u_k(x) \neq 0,\ s_\ell(x) > 0\},$$

where $f_i = \operatorname{num} \beta^*(\check{f}_i)$, $p_j = \operatorname{num} \beta^*(\check{p}_j)$ *and* $u_k = \operatorname{num} \beta^*(\check{u}_k)$.

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- Useful to detect structure identifiability in graphical models.

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- The model is contained in PD_V and its vanishing ideal is the saturation of $CI_G = \langle |\Sigma_{ij|[j-1]\setminus i}| : ij \notin E(G)\rangle$ at the leading principal minors.

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Yields constraints of the sort $|\Sigma_{[l-1]}||\Sigma_{ij|[j-1]\setminus i}| = |\Sigma_{[k-1]}||\Sigma_{kl|[l-1]\setminus k}|$ [BKMS24].

Hard implicitizations become easy

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-- Vanishing ideal via built-in elimination method:
time I1 = gaussianVanishingIdeal R;

-- Vanishing ideal via saturation:
time (
  prs = for i in V list (
    P := toList parents(G, i);
    if #P == 0 then 1 else det submatrix(S, P, P)
  );
  J = ideal for ij in toList(allE-set(edges G)) list (
    P := toList parents(G, ij#1);
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  I2 = fold(saturate, J, prs);
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I1 == I2 --> true
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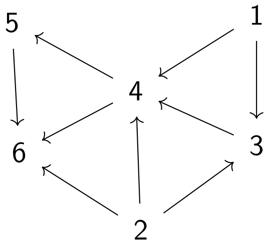

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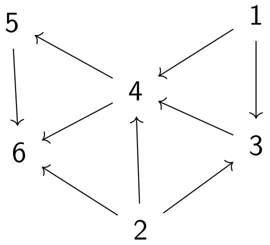

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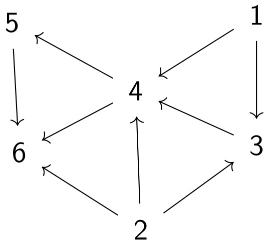

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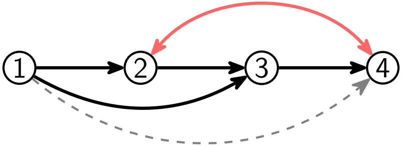

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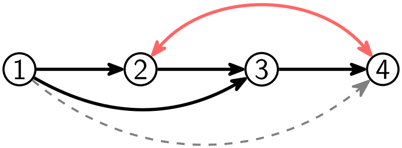

$$\omega_{11} = |\Sigma_1|,\ \omega_{22} = \frac{|\Sigma_{12}|}{|\Sigma_1|},\ \omega_{33} = \frac{|\Sigma_{123}|}{|\Sigma_{12}|},\ \omega_{44} = \frac{|\Sigma_{1234}|}{|\Sigma_{123}|} + \frac{|\Sigma_{12}||\Sigma_{24|13}|^2}{|\Sigma_1||\Sigma_{123}|^2},$$

$$\omega_{24} = \frac{|\Sigma_{12}||\Sigma_{24|13}|}{|\Sigma_1||\Sigma_{123}|},\ \lambda_{12} = \frac{|\Sigma_{12|\emptyset}|}{|\Sigma_1|},\ \lambda_{13} = \frac{|\Sigma_{13|2}|}{|\Sigma_{12}|},\ \lambda_{23} = \frac{|\Sigma_{23|1}|}{|\Sigma_{12}|},\ \lambda_{34} = \frac{|\Sigma_{34|12}|}{|\Sigma_{123}|},$$

$$\lambda_{14} = \frac{|\Sigma_1||\Sigma_{14|23}| + |\Sigma_{12|\emptyset}||\Sigma_{24|13}|}{|\Sigma_1||\Sigma_{123}|}$$

Examples: Continuous Lyapunov models

- Can also compute vanishing ideals for Lyapunov models, e.g. $1 \longrightarrow 2 \longrightarrow 3$:

$$\sigma_{11}\sigma_{12}^2\sigma_{13}\sigma_{22} - \sigma_{11}^2\sigma_{13}\sigma_{22}^2 - \sigma_{11}\sigma_{12}^3\sigma_{23} + \sigma_{11}\sigma_{12}\sigma_{13}^2\sigma_{23} + \sigma_{11}^2\sigma_{12}\sigma_{22}\sigma_{23} + \sigma_{12}\sigma_{13}^2\sigma_{22}\sigma_{23} - \sigma_{11}^2\sigma_{13}\sigma_{23}^2 - 2\sigma_{12}^2\sigma_{13}\sigma_{23}^2 + \sigma_{11}\sigma_{13}\sigma_{22}\sigma_{23}^2 - \sigma_{11}\sigma_{12}^2\sigma_{13}\sigma_{33} - \sigma_{11}\sigma_{13}\sigma_{22}^2\sigma_{33} + \sigma_{11}^2\sigma_{12}\sigma_{23}\sigma_{33} + \sigma_{12}^3\sigma_{23}\sigma_{33} = 0.$$

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- This irreducible quintic specializes to $\sigma_{13} = \sigma_{12}\sigma_{23}$ when $\sigma_{11} = \sigma_{22} = \sigma_{33} = 1$ which also happens to cut out the Bayesian network model of $1 \longrightarrow 2 \longrightarrow 3 \ldots$

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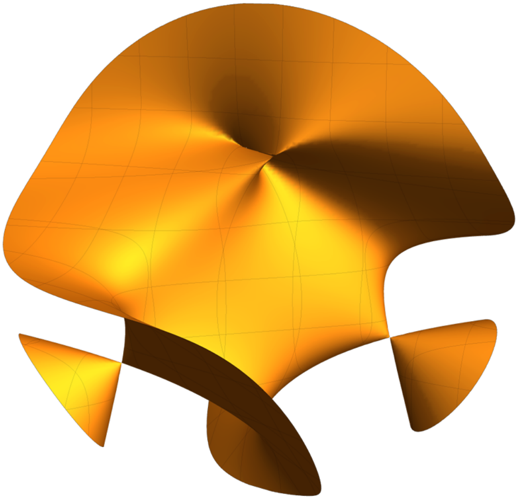
A combinatorial "separation" criterion is not yet known.

An edge-colored Lyapunov model



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