On the Intersection and Composition properties for discrete random variables

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- The CI symbols are symmetric $[I \perp\!\!\!\perp J \mid K] \iff [J \perp\!\!\!\perp I \mid K]$.
- A set S of CI symbols is a semigraphoid if it satisfies

$$\begin{aligned}[I \perp\!\!\!\perp JK \mid L] &\iff [I \perp\!\!\!\perp J \mid L] \wedge [I \perp\!\!\!\perp K \mid JL] \\ &\iff [I \perp\!\!\!\perp K \mid L] \wedge [I \perp\!\!\!\perp J \mid KL]\end{aligned}$$

- E.g., conditional independence relation of every system of random variables.

Partial converses of the semigraphoid property

$$[I \perp\!\!\!\perp JK \mid L] \implies \begin{cases} ①[I \perp\!\!\!\perp J \mid L] \wedge ②[I \perp\!\!\!\perp J \mid KL] \wedge \\ ③[I \perp\!\!\!\perp K \mid L] \wedge ④[I \perp\!\!\!\perp K \mid JL] \end{cases}$$

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Modulo the semigraphoid axioms Intersection and Composition are logical converses:

Intersection	② ∧ ④	$\implies$	① ∧ ③
Composition	② ∧ ④	$\impliedby$	① ∧ ③

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- Curiously they are dual to each other via $[I \perp\!\!\!\perp J \mid K]^* := [I \perp\!\!\!\perp J \mid N \setminus IJK]$:

Intersection $\quad [I \perp\!\!\!\perp J \mid KL] \wedge [I \perp\!\!\!\perp K \mid JL] \implies [I \perp\!\!\!\perp J \mid L] \wedge [I \perp\!\!\!\perp K \mid L]$

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$$\begin{array}{lc} \text{Intersection} & [I \perp\!\!\!\perp J \mid KL] \wedge [I \perp\!\!\!\perp K \mid JL] \implies [I \perp\!\!\!\perp J \mid L] \wedge [I \perp\!\!\!\perp K \mid L] \\ \downarrow & \text{use } \bar{L} = N \setminus IJKL \\ \text{Intersection}^* & [I \perp\!\!\!\perp J \mid \bar{L}] \wedge [I \perp\!\!\!\perp K \mid \bar{L}] \implies [I \perp\!\!\!\perp J \mid K\bar{L}] \wedge [I \perp\!\!\!\perp K \mid J\bar{L}] \end{array}$$

but this is Composition with L replaced by $\bar{L}$.

Examples

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  - d-separation, u-separation, m-separation, $*$-separation, ...
- Positive distributions satisfy Intersection.
- MTP_2 distributions satisfy Composition.

Non-example: matroids

- Let $r\colon 2^N \to \mathbb{Z}$ be a matroid. The set of modular pairs of r is a semigraphoid:

$$\mathscr{S}(r) := \{\, [I \perp\!\!\!\perp J \mid K] : r(IK) + r(JK) = r(IJK) + r(K) \,\}.$$

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Lemma

If S satisfies Composition and $[i \perp\!\!\!\perp j]$ for all $i \neq j$ then S is totally independent.

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Lemma*

If S satisfies Intersection and $[i \perp\!\!\!\perp j \mid N \setminus ij]$ for all $i \neq j$ then S is totally independent.

- *If r is a co-simple matroid then $\mathscr{S}(r)$ satisfies Intersection if and only if r is zero.*

Intersection for three binary random variables

$$[I \perp\!\!\!\perp J \mid KL] \wedge [I \perp\!\!\!\perp K \mid JL] \implies [I \perp\!\!\!\perp J \mid L] \wedge [I \perp\!\!\!\perp K \mid L]$$

- By marginalizing to $IJKL$, conditioning on L and viewing I, J, K as single random variables, we can reduce one instance of Intersection to the trivariate case.

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$$\langle p_{110}, p_{101}, p_{010}, p_{001} \rangle \cap \langle p_{111}, p_{100}, p_{011}, p_{000} \rangle$$

- Failure of Intersection only on the boundary. Full support implies Intersection.

The characteristic bipartite graph

- Let i, j, k be discrete random variables taking r_i, r_j, r_k states, respectively.
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Theorem (Cartwright–Engström conjecture & Fink's theorem [Fin11])

The conditional independence model $\mathscr{M}([i \perp\!\!\!\perp j \mid k] \wedge [i \perp\!\!\!\perp k \mid j])$ decomposes into irreducible components, one for each admissible bipartite graph on $[r_j] \sqcup [r_k]$. Only the component corresponding to K_{r_j, r_k} is fully contained in $\mathscr{M}([i \perp\!\!\!\perp jk])$.

- Hence, $G(j, k)$ being connected is sufficient for (one instance of) Intersection.

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- Also known as the Double Markov property [CK11, Exercise 16.25].

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- What is necessary to construct such a g à la Gács–Körner?

Conditional Ingleton vs. Gács–Körner

It is not difficult to parametrize binary distributions which satisfy the conditional Ingleton criterion but fail the common information criterion using Cylindrical Algebraic Decomposition in `Mathematica`, e.g., having $G(j,k) = \{0{-}1, 1{-}0\}$.

i	j	k	g	Pr
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- Distribution on ijk is quasi-uniform and $[i \perp\!\!\!\perp jk]$ holds.

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There is only one irreducible component of $\mathscr{M}([i \perp\!\!\!\perp j] \wedge [i \perp\!\!\!\perp k])$ on which the sum of all probabilities does not vanish.

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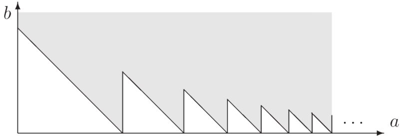
- No graphs, no interesting boundary structure.
- There exist positive distributions violating Composition.

Matúš's criterion

- A distribution on N is tight if each $i \in N$ functionally depends on $N \setminus i$.

Theorem ([Mat06])

The tight entropy profiles with $[i \perp\!\!\!\perp j]$ and $[i \perp\!\!\!\perp k]$ are described by a piecewise linear information inequality $\rightarrow$



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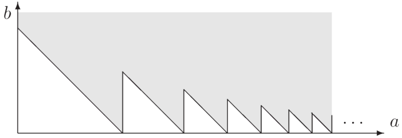

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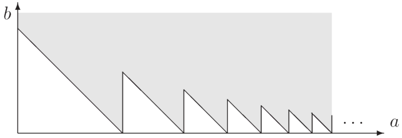

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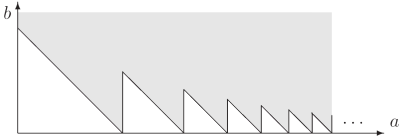

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- How to generalize away from the tightness constraints?

Dual conditional Ingleton criterion

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The following is an essentially conditional information inequality:

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- How to use this? Any constructions of suitable g?

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- There are 1 098 probabilistically representable semigraphoids (up to permutation).

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- Is there a (geometric / algebraic) relation between Intersection and Composition on the level of distributions?

References I

[AAZ22] Arash A. Amini, Bryon Aragam, and Qing Zhou. *A non-graphical representation of conditional independence via the neighbourhood lattice*. 2022. arXiv: 2206.05829 [math.ST].

[CK11] Imre Csiszár and János Körner. *Information theory. Coding theorems for discrete memoryless systems*. 2nd ed. Cambridge University Press, 2011. DOI: 10.1017/CBO9780511921889.

[Fin11] Alex Fink. "The binomial ideal of the intersection axiom for conditional probabilities". In: *J. Algebr. Comb.* 33.3 (2011), pp. 455–463. DOI: 10.1007/s10801-010-0253-5.

[GK73] Peter Gács and János Körner. "Common information is far less than mutual information". In: *Probl. Control Inf. Theory* 2 (1973), pp. 149–162.

[KRS19] Thomas Kahle, Johannes Rauh, and Seth Sullivant. "Algebraic aspects of conditional independence and graphical models". In: *Handbook of graphical models*. Ed. by Marloes Maathuis, Mathias Drton, Steffen Lauritzen, and Martin Wainwright. Chapman & Hall/CRC Handbooks of Modern Statistical Methods. CRC Press, 2019, pp. 61–80. ISBN: 978-1-4987-8862-5; 978-0-4298-7424-6.

References II

[Kir07] George A. Kirkup. "Random variables with completely independent subcollections". In: *J. Algebra* 309.2 (2007), pp. 427–454. DOI: 10.1016/j.jalgebra.2006.06.023.

[MMR05] Ernesto San Martín, Michel Mouchart, and Jean-Marie Rolin. "Ignorable common information, null sets and Basu's first theorem". In: *Sankhyā* 67.4 (2005), pp. 674–698.

[Mat06] František Matúš. "Piecewise linear conditional information inequality". In: *IEEE Trans. Inf. Theory* 52.1 (2006), pp. 236–238. DOI: 10.1109/TIT.2005.860438.

[Šim06] Petr Šimeček. "A short note on discrete representability of independence models". In: *Proccedings of the European Workshop on Probabilistic Graphical Models*. 2006, pp. 287–292.

[Stu05] Milan Studený. *Probabilistic Conditional Independence Structures*. Information Science and Statistics. Springer, 2005.

[Stu21] Milan Studený. "Conditional independence structures over four discrete random variables revisited: conditional ingleton inequalities". In: *IEEE Trans. Inf. Theory* 67.11 (2021), pp. 7030–7049. DOI: 10.1109/TIT.2021.3104250.