WISST! Galois connections

A *partial order* on a set X is a relation $\leq$ which satisfies

$$x \leq x, \qquad \text{(Reflexivity)}$$
$$x \leq x' \wedge x' \leq x'' \Rightarrow x \leq x'', \qquad \text{(Transitivity)}$$
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A partially ordered set in which each subset A has a *least upper bound* $\bigvee A$ and a *greatest lower bound* $\bigwedge A$ is a *complete lattice*.

Galois connections

A *closure operator* on $(X, \leq)$ is a function $\mathfrak{c} : X \to X$ with

$$\mathfrak{c}(x) \geq x, \qquad \text{(Extensivity)}$$

$$x \leq x' \Rightarrow \mathfrak{c}(x) \leq \mathfrak{c}(x'), \qquad \text{(Monotonicity)}$$

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A *Galois connection* between $(X, \leq)$ and $(Y, \leq)$ is a pair of antitone maps $\alpha : X \to Y$ and $\beta : Y \to X$ such that $\beta\alpha$ and $\alpha\beta$ are closure operators on X and Y, respectively.

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$$\begin{aligned} \mathfrak{c}(x) &\geq x, && \text{(Extensivity)} \\ x \leq x' &\Rightarrow \mathfrak{c}(x) \leq \mathfrak{c}(x'), && \text{(Monotonicity)} \\ \mathfrak{c}(\mathfrak{c}(x)) &= \mathfrak{c}(x). && \text{(Idempotency)} \end{aligned}$$

A *Galois connection* between $(X, \leq)$ and $(Y, \leq)$ is a pair of antitone maps $\alpha : X \to Y$ and $\beta : Y \to X$ such that $\beta\alpha$ and $\alpha\beta$ are closure operators on X and Y, respectively.

- Every topology is defined by a closure operator.
- The closed sets form a complete lattice.
- Between the closed sets of X and Y, the Galois connection establishes a *lattice antiisomorphism*.

Formal concept analysis

Let $\mathcal{O}$ be a set of *objects* and $\mathcal{A}$ a set of *attributes* with an *incidence* relation $\diamond$. This defines a Galois connection between the powersets of $\mathcal{O}$ and $\mathcal{A}$:

$$\alpha(O) := \{\, a \in \mathcal{A} : o \diamond a \;\; \forall o \in O \,\},$$
$$\beta(A) := \{\, o \in \mathcal{O} : o \diamond a \;\; \forall a \in A \,\}.$$

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The closure operator $\beta\alpha$ *saturates* a set of objects O with respect to all of its attributes.

The other closure operator $\alpha\beta$ *infers* all attributes which are implied on $\mathcal{O}$ by a set of attributes A.

Example: Galois theory

Let L/F be a finite field extension with $G = \text{Aut}(L/F)$. Then the maps

$$L/K/F \quad \longleftrightarrow \quad 1 \leq H \leq G,$$

given by "fixgroup" and "fixed field" are a Galois connection between intermediate fields in L/F and subgroups of G.

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This connection is defined by the relation $x \diamond g :\Leftrightarrow g(x) = x$.

Example: Algebraic geometry

For $\mathbb{K}$ algebraically closed, consider the relation

$$a \diamond f \ :\Leftrightarrow\ f(a) = 0$$

on $\mathbb{K}^n$ (objects) and $\mathbb{K}[x_1, \ldots, x_n]$ (attributes).

The closed object sets are the algebraic varieties and the closed attribute sets are the radical ideals, by Hilbert's Nullstellensatz.

Example: Convex geometry

Between $\mathbb{R}^n$ and $(\mathbb{R}^n)^*$ define

$$x \diamond \alpha \;:\Leftrightarrow\; \alpha(x) \geq 0.$$

This gives a Galois connection whose closed object sets are the closed convex cones, by the separation lemma of convex sets.

The dual closed sets are closed convex cones as well, in $(\mathbb{R}^n)^*$. The Galois connection constructs the *dual cone* in both directions.

Example #4

Let $\mathcal{K} = \{\, g_i \geq 0 \,\}$ be a semialgebraic set. Consider the following relation between subsets of $\mathcal{K}$ and subets of $\{\, g_i \,\}$:

$$x \diamond g \;:\Leftrightarrow\; g(x) = 0.$$

What is the name of this construction? If $\mathcal{K}$ is a polyhedron, then its closed subsets form the *face lattice*.