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Two universality results for Gaussian CI models

Magdeburg, April 28, 2021

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DFG-Graduiertenkolleg
MATHEMATISCHE
KOMPLEXITÄTSREDUKTION

Gaussian conditional independence

Consider random variables $(\xi_i)_{i \in N} \sim \mathcal{N}(\mu, \Sigma)$. The *conditional independence (CI) statement* $\xi_i \perp\!\!\!\perp \xi_j \mid \xi_K$ conveys, informally, that if ξ_K is known, then learning the value of one variable does not give any information about the other one.

Definition

The polynomial $\Sigma[K] := \det \Sigma_{K,K}$ is a *principal minor* of Σ and $\Sigma[ij|K] := \det \Sigma_{iK,jK}$ is an *almost-principal minor*.

If Σ is positive-definite, then $\Sigma[K] > 0$, and $\xi_i \perp\!\!\!\perp \xi_j \mid \xi_K$ holds if and only if $\Sigma[ij|K] = 0$.



Almost-principal minors

$$\Sigma[ij] = x_{ij}$$

$$\Sigma[ij|k] = x_{ij}x_{kk} - x_{ik}x_{jk}$$

$$\Sigma[ij|kl] = x_{ij}x_{kk}x_{ll} - x_{il}x_{jl}x_{kk} + x_{il}x_{jk}x_{kl} + x_{ik}x_{jl}x_{kl} - x_{ij}x_{kl}^2 - x_{ik}x_{jk}x_{ll}$$

$$\begin{aligned}\Sigma[ij|klm] = & x_{ij}x_{kk}x_{ll}x_{mm} + x_{im}x_{jm}x_{kl}^2 - x_{im}x_{jl}x_{kl}x_{km} - x_{il}x_{jm}x_{kl}x_{km} + x_{il}x_{jl}x_{km}^2 \\ & - x_{im}x_{jm}x_{kk}x_{ll} + x_{im}x_{jk}x_{km}x_{ll} + x_{ik}x_{jm}x_{km}x_{ll} - x_{ij}x_{km}^2x_{ll} \\ & + x_{im}x_{jl}x_{kk}x_{lm} + x_{il}x_{jm}x_{kk}x_{lm} - x_{im}x_{jk}x_{kl}x_{lm} - x_{ik}x_{jm}x_{kl}x_{lm} \\ & - x_{il}x_{jk}x_{km}x_{lm} - x_{ik}x_{jl}x_{km}x_{lm} + 2x_{ij}x_{kl}x_{km}x_{lm} + x_{ik}x_{jk}x_{lm}^2 \\ & - x_{ij}x_{kk}x_{lm}^2 - x_{il}x_{jl}x_{kk}x_{mm} + x_{il}x_{jk}x_{kl}x_{mm} + x_{ik}x_{jl}x_{kl}x_{mm} \\ & - x_{ij}x_{kl}^2x_{mm} - x_{ik}x_{jk}x_{ll}x_{mm}\end{aligned}$$

⋮



Models and inference

Definition

A *CI constraint* is a CI statement $\xi_i \perp\!\!\!\perp \xi_j \mid \xi_K$ or its negation $\neg(\xi_i \perp\!\!\!\perp \xi_j \mid \xi_K)$. They are *algebraic conditions* on the entries of Σ , equivalent to vanishing or non-vanishing of the almost-principal minors $\Sigma[ij|K]$.

Definition

The *model* of a set of CI constraints is the set of all positive-definite matrices which satisfy the constraints.



Models and inference

Consider two sets of CI statements \mathcal{L} and \mathcal{M} :

$$\bigwedge \mathcal{L} \Rightarrow \bigvee \mathcal{M}$$



Models and inference

Consider two sets of CI statements \mathcal{L} and \mathcal{M} :

$$\begin{array}{ccc} \bigwedge \mathcal{L} \Rightarrow \bigvee \mathcal{M} & \iff & \mathcal{L} \cup \neg \mathcal{M} \\ \text{is not valid} & & \text{has a model} \end{array}$$



















































Models and inference









































Consider two sets of CI statements \mathcal{L} and \mathcal{M} :

$$\begin{array}{ccc} \bigwedge \mathcal{L} \Rightarrow \bigvee \mathcal{M} & \iff & \mathcal{L} \cup \neg \mathcal{M} \\ \text{is not valid} & & \text{has a model} \end{array}$$

Reasoning about relevance statements in normally distributed random variables is **the same** as reasoning about the vanishing of very special kinds of determinants in the positive-definite matrices.



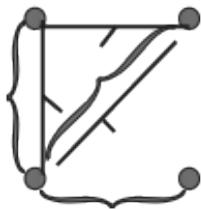
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 Model M7 4 0 -2 -2 0 4 -3 1 -2 -3 4 -1 2 1 -1 4	 Model M8 4 1 -2 -2 1 4 -3 2 -2 -3 4 -1 2 2 -1 4	 Model M9 4 2 -3 3 2 4 -3 1 -3 -3 4 -2 3 1 -2 4	 Model M10 4 1 -2 -2 1 4 -3 2 -2 -3 4 -1 2 2 -2 4	 Model M11 4 0 -3 -3 0 4 1 2 -3 1 4 2 -3 2 2 4	 Model M12 4 0 -2 -1 0 4 -3 -2 -2 -3 4 0 1 -2 0 4
 Model M13 4 1 1 2 -2 -1 4 -2 -2 2 -4 -3 -2 -2 2 -3 4	 Model M14 4 1 2 -2 1 4 -1 -2 -2 1 4 -2 -2 -2 -2 4	 Model M15 6 1 4 -2 1 6 -5 3 -4 -5 6 -3 2 3 -3 6	 Model M16 4 0 2 -1 0 4 2 -3 2 2 4 -3 -1 -3 -3 4	 Model M17 4 0 -2 1 0 4 -3 2 -2 -3 4 -2 1 2 -2 4	 Model M18 4 0 -2 2 0 4 -3 -1 -2 3 4 -1 2 -1 -1 4
 Model M19 4 0 2 -2 0 4 -1 -3 2 -1 4 0 -2 -3 0 4	 Model M20 4 0 -2 -3 0 4 2 -1 -2 2 4 -1 -3 -1 1 4	 Model M21 4 0 -2 2 0 4 -2 -2 -2 2 4 -1 2 -2 -1 4	 Model M22 4 0 -2 2 0 4 -2 2 -2 2 4 -1 2 2 -1 4	 Model M23 4 0 2 -1 0 4 -1 2 2 1 4 2 -1 -2 2 4	 Model M24 6 0 -3 1 0 6 4 -3 -3 4 -6 2 1 3 -2 6
 Model M25 8 0 4 -3 0 8 4 -7 4 4 8 -6 -3 -7 -6 8	 Model M26 8 -3 -6 -6 -3 8 4 4 -6 4 8 2 -6 4 2 8	 Model M27 8 0 4 -2 0 8 2 -5 4 2 8 4 -2 -5 -8 8	 Model M28 10 -4 -5 -8 -4 10 2 5 -5 2 10 1 -8 5 11 0	 Model M29 10 -4 -8 -5 -4 10 2 8 -8 2 10 4 -8 4 10 5	 Model M30 4 0 -2 -2 0 4 -1 -1 -2 1 4 0 -2 -1 0 4
 Model M31 12 -3 -6 -2 -3 12 6 8 -6 6 12 8 -2 8 8 12	 Model M32 20 5 -10 10 5 20 -10 10 -10 -10 20 -8 10 10 -8 20	 Model M33 4 0 -3 -3 0 4 0 1 -3 0 4 1 -3 1 1 4	 Model M34 4 1 -3 -2 -1 4 1 2 -3 1 4 2 -2 2 2 4	 Model M35 4 0 -2 -3 0 4 3 2 -3 3 4 0 -3 2 0 4	 Model M36 4 1 -2 2 1 4 -2 2 -2 2 4 -1 2 2 -1 4
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 Model M77 2 0 1 0 0 2 1 -2 1 1 2 -1 0 -2 1 2	 Model M78 -1 4 -2 -2 1 4 -2 2 2 2 4 4 -2 2 4 4	 Model M79 5 0 4 0 0 5 3 5 4 3 5 3 0 -5 3 5	 Model M80 2 -1 -2 -1 -1 2 1 2 -2 1 2 1 -1 2 1 2	 Model M81 2 -2 -1 2 -2 2 1 2 -1 1 2 1 -2 2 1 2	 Model M82 2 0 0 0 0 2 2 1 0 2 2 1 0 1 2 1	
 Model M83 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1	 Model M84 1 0 0 0 0 1 1 1 0 1 1 1 0 1 1 1	 Model M85 1 a b c a 1 d e b d 1 f c e f 1	$\text{When } \begin{cases} a = \frac{1}{\sqrt{107463}} \\ b = \frac{1}{\sqrt{63236}} \\ c = \frac{1}{\sqrt{107463}} \\ d = 10e = \frac{100}{\sqrt{107463}} \\ f = 10e = \frac{1}{4} \cdot \frac{1}{10} \end{cases}$			
 Model M86  Model M87  Model M88						

Petr Šimeček. Gaussian representation of independence models over four random variables.
In *COMPSTAT* conference, 2006.



Model M85



1	a	b	c
a	1	d	e
b	d	1	f
c	e	f	1

Where:

$$a = \frac{3}{632836} \sqrt{1107463},$$

$$b = 10c = \frac{100}{158209} \sqrt{1107463}$$

$$d = 10e = \frac{3}{4}, f = \frac{1}{10}$$



Šimeček's Question

Does every non-empty Gaussian CI model contain a rational point?



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Model M85

Where:

$$a = \frac{3}{632836} \sqrt{1107463},$$

$$b = 10c = \frac{100}{158209} \sqrt{1107463}$$

$$d = 10e = \frac{3}{4}, f = \frac{1}{10}$$



1	a	b	c
a	1	d	e
b	d	1	f
c	e	f	1

$$\begin{pmatrix} 1 & -1/17 & -49/51 & -7/17 \\ -1/17 & 1 & 1/3 & 1/7 \\ -49/51 & 1/3 & 1 & 3/7 \\ -7/17 & 1/7 & 3/7 & 1 \end{pmatrix}$$



Complexity bounds

Let $f_1, \dots, f_r \in \mathbb{Z}[t_1, \dots, t_k]$ be integer polynomials in finitely many variables. We consider a system of polynomial constraints “ $f_i \bowtie 0$ ” where $\bowtie \in \{=, \neq, <, \leq, \geq, >\}$.

Theorem (Tarski’s transfer principle)

If a polynomial system $\{f_i \bowtie 0\}$ has a solution over \mathbb{R} , then it has a solution in a finite real extension of \mathbb{Q} .

Theorem (Real Nullstellensatz)

A polynomial F vanishes on the semialgebraic set $\mathcal{K} = \{f_i \bowtie 0\}$ if and only if $F \in \sqrt[\mathbb{R}]{\mathcal{I}(f_i \bowtie 0)}$. The ideal $\mathcal{I}(f_i \bowtie 0)$, its real radical and the membership of F can be computed.

Keyword for this decision problem: “existential theory of the reals”.



Main results

Theorem

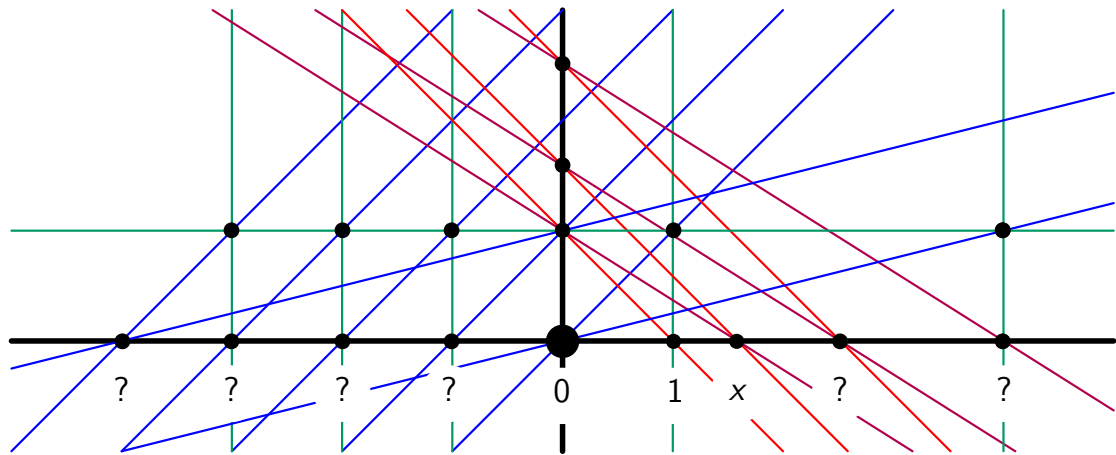
Let $d \geq 1$ and $\mathbb{Q}^{(d)}$ the field generated by all real algebraic numbers of degree at most d . For every d there exists a non-empty Gaussian CI model which has no $\mathbb{Q}^{(d)}$ -rational point.

Theorem

For every system of polynomials defining a semialgebraic set $\mathcal{K} = \{f_i \geq 0\}$ there exists a Gaussian CI model which is inhabited over \mathbb{R} if and only if \mathcal{K} is non-empty. Moreover, the description of this model is polynomially-sized in the description of \mathcal{K} .



Breakout riddle

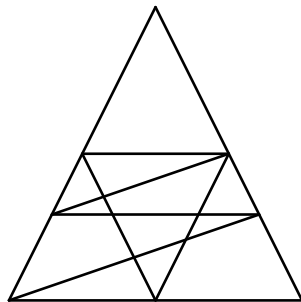


Algebra \subseteq Synthetic geometry

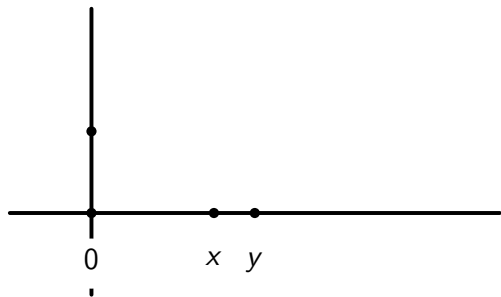
Point and line configuration for the equation $x^2 - 2 = 0$.

The configuration is specified by incidences between points and lines and also the parallelities of lines.

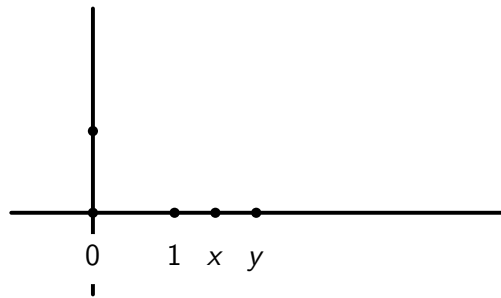
It is realizable over $\mathbb{Q}(\sqrt{2})$ but not over \mathbb{Q} .



Von Staudt constructions



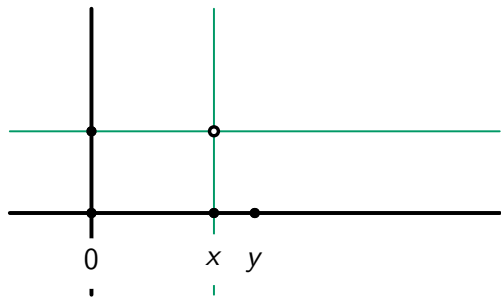
Addition



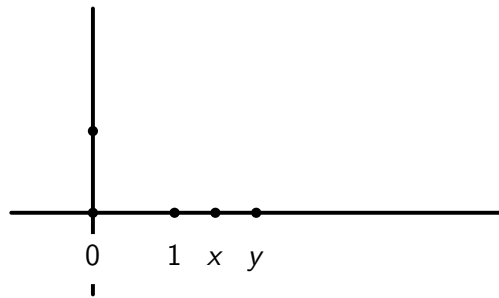
Multiplication



Von Staudt constructions



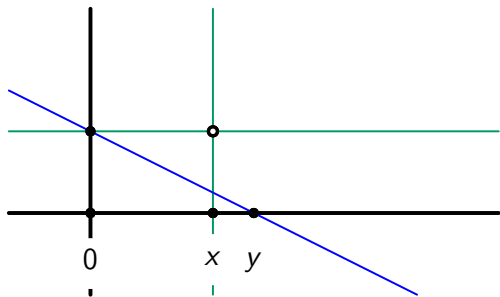
Addition



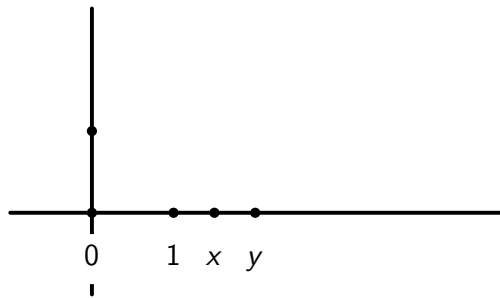
Multiplication



Von Staudt constructions



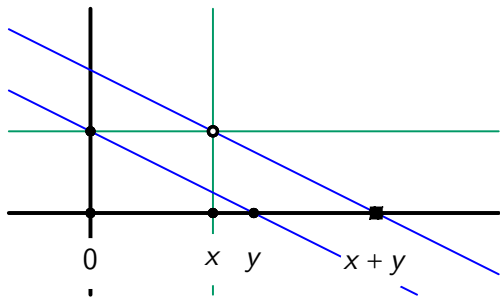
Addition



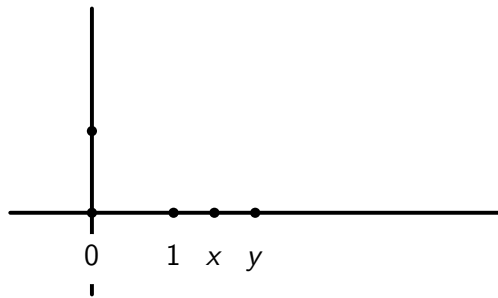
Multiplication



Von Staudt constructions



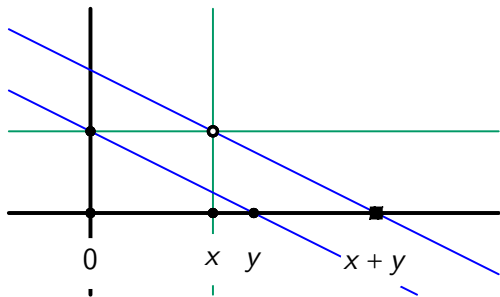
Addition



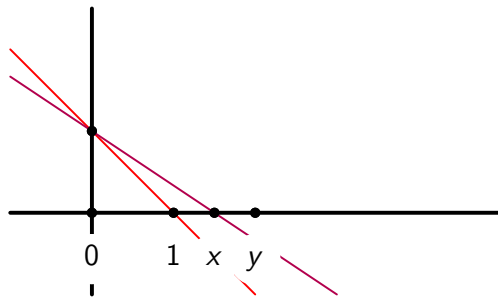
Multiplication



Von Staudt constructions



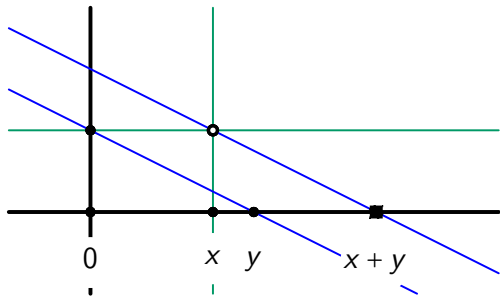
Addition



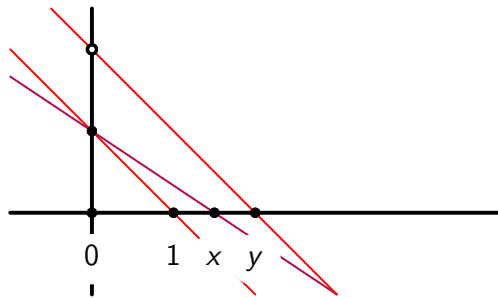
Multiplication



Von Staudt constructions



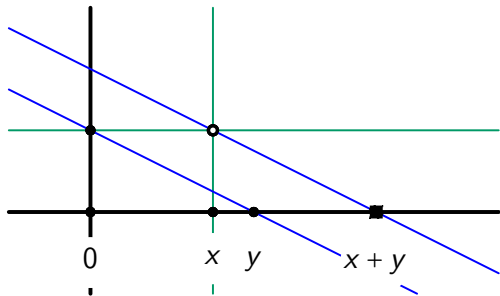
Addition



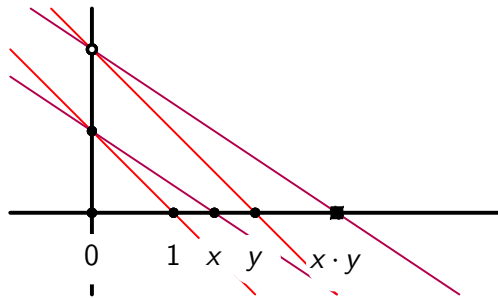
Multiplication



Von Staudt constructions



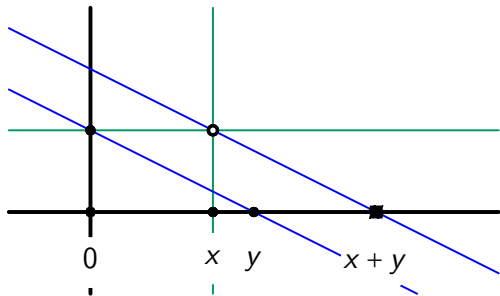
Addition



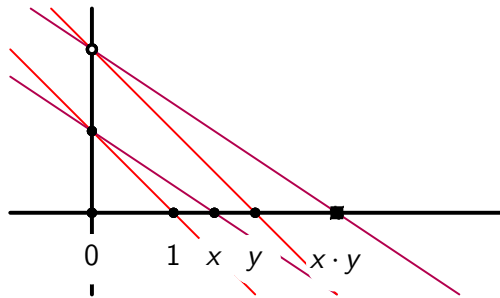
Multiplication



Von Staudt constructions



Addition



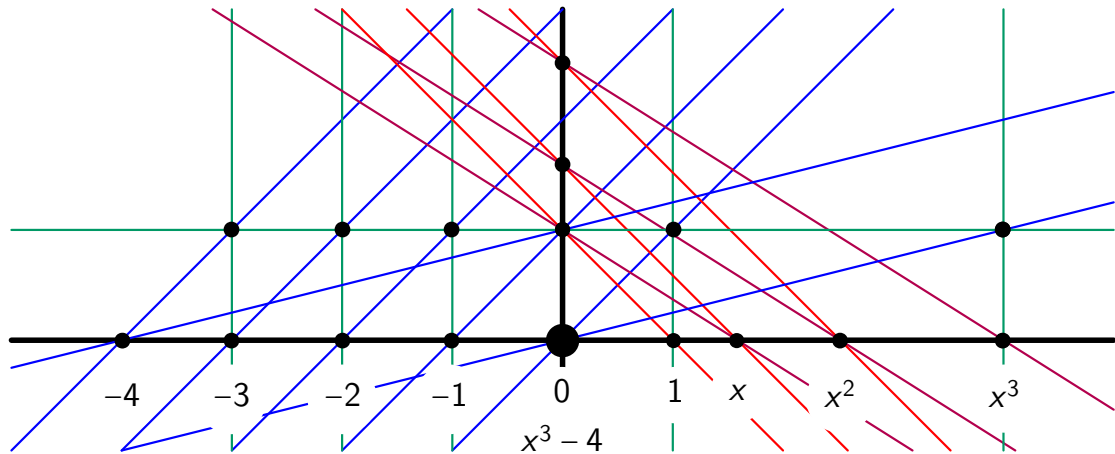
Multiplication

Lemma

The evaluation of integer polynomials can be encoded with incidence geometry.



The cube root of 4



Incidence relations as conditional independence

Suppose $E = \{x, y, z\}$ and Σ_E is the identity matrix.

$$\Sigma[ij|E] = \Sigma[E] (\Sigma_{ij} - \Sigma_{i,E} \Sigma_E^{-1} \Sigma_{E,j}) \quad (\triangleleft)$$

$$\left. \begin{array}{l} \Sigma[ij|E] = 0 \Leftrightarrow \Sigma_{ij} = \langle \Sigma_{i,E}, \Sigma_{j,E} \rangle \\ \Sigma[ij] = 0 \Leftrightarrow \Sigma_{ij} = 0 \end{array} \right\} \Leftrightarrow \Sigma_{i,E} \perp \Sigma_{j,E}$$



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$p = \Sigma_{i,E} = [p_x : p_y : p_z]$ and $\ell = \Sigma_{j,E} = [\ell_x : \ell_y : \ell_z]$ are the *homogeneous coordinates* of a point and a line in the projective plane with $p \in \ell^\perp$.



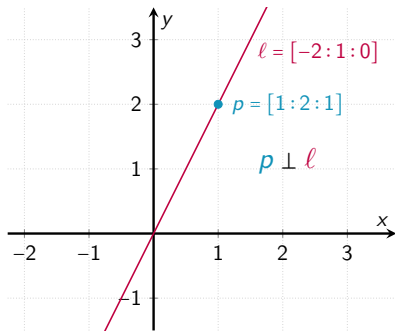
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$p = \Sigma_{i,E} = [p_x : p_y : p_z]$ and $l = \Sigma_{j,E} = [l_x : l_y : l_z]$ are the *homogeneous coordinates* of a point and a line in the projective plane with $p \in l^\perp$.



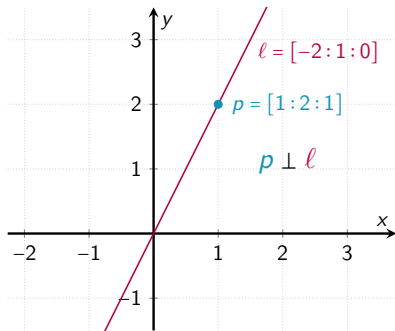
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Lemma

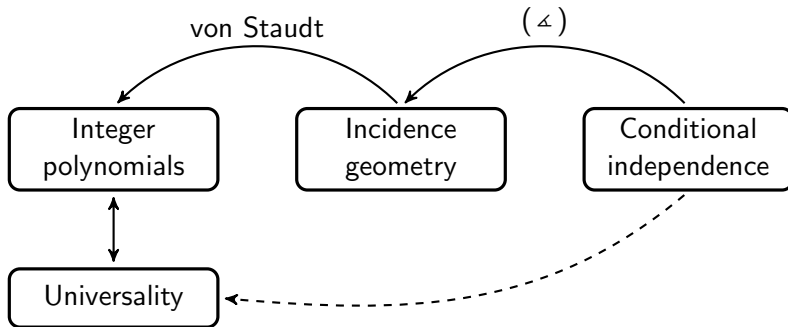
Incidence geometry can be encoded in CI constraints.



Polynomial evaluation as conditional independence

$$\begin{array}{c}
 p_1 \\
 \vdots \\
 p_n \\
 l_1 \\
 \vdots \\
 l_m \\
 x \\
 y \\
 z
 \end{array}
 \left(
 \begin{array}{ccc|ccc|ccc}
 p_1 & \dots & p_n & l_1 & \dots & l_m & x & y & z \\
 p_1^* & & \langle p, p' \rangle & & & & p_1^x & p_1^y & p_1^z \\
 & \ddots & & & \langle p, l \rangle & & & \vdots & \\
 \langle p', p \rangle & & p_n^* & & & & p_n^x & p_n^y & p_n^z \\
 \hline
 & & & l_1^* & & \langle l, l' \rangle & l_1^x & l_1^y & l_1^z \\
 & \langle l, p \rangle & & & \ddots & & & \vdots & \\
 \hline
 & & & \langle l', l \rangle & & l_m^* & l_m^x & l_m^y & l_m^z \\
 \hline
 p_1^x & & p_n^x & l_1^x & & l_m^x & 1 & 0 & 0 \\
 p_1^y & \dots & p_n^y & l_1^y & \dots & l_m^y & 0 & 1 & 0 \\
 p_1^z & & p_n^z & l_1^z & & l_m^z & 0 & 0 & 1
 \end{array}
 \right)$$





Theorem

To every polynomial system $\{f_i \approx 0\}$ there is a set of CI constraints which has a model over a field \mathbb{K}/\mathbb{Q} if and only if the polynomial system has a solution in \mathbb{K} .



Theorem

Let $d \geq 1$ and $\mathbb{Q}^{(d)}$ the field generated by all real algebraic numbers of degree at most d . For every d there exists a non-empty Gaussian CI model which has no $\mathbb{Q}^{(d)}$ -rational point.

Theorem

For every system of polynomials defining a semialgebraic set $\mathcal{K} = \{f_i \bowtie 0\}$ there exists a Gaussian CI model which is inhabited over \mathbb{R} if and only if \mathcal{K} is non-empty. Moreover, the description of this model is polynomially-sized in the description of \mathcal{K} .

Question: What is the smallest n (≥ 5) for which there is an n -variate Gaussian CI model without rational point?





Tobias Boege.

Incidence geometry in the projective plane via almost-principal minors of symmetric matrices, 2021.

[arXiv:2103.02589](https://arxiv.org/abs/2103.02589).



Jürgen Bokowski and Bernd Sturmfels.

Computational synthetic geometry, volume 1355 of *Lecture Notes in Mathematics*. Springer, 1989.



Jürgen Richter-Gebert.

Perspectives on projective geometry. A guided tour through real and complex geometry. Springer, 2011.



Petr Šimeček.

Gaussian representation of independence models over four random variables.
In *COMPSTAT conference*, 2006.



$\bowtie \in \{=\}$ suffices

In the real numbers, the solvability of a system of equations is just as hard as equations, inequations and inequalities if we introduce a new variable y :

- $f(x) \neq 0 \Leftrightarrow \exists y : yf(x) - 1 = 0$ ($f(x)$ has a multiplicative inverse)
- $f(x) \geq 0 \Leftrightarrow \exists y : f(x) - y^2 = 0$ ($f(x)$ is a square, i.e. non-negative)
- $f(x) > 0 \Leftrightarrow \exists y : y^2 f(x) - 1 = 0$ ($f(x)$ has an inverse which is a square)

