Tobias Boege

Two universality results for Gaussian CI models

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DFG-Graduiertenkolleg
MATHEMATISCHE
KOMPLEXITÄTSREDUKTION

Gaussian conditional independence

Consider random variables $(\xi_i)_{i \in N} \sim \mathcal{N}(\mu, \Sigma)$. The *conditional independence (CI) statement* $\xi_i \perp\!\!\!\perp \xi_j \mid \xi_K$ conveys, informally, that if ξ_K is known, then learning the value of one variable does not give any information about the other one.

Definition

The polynomial $\Sigma[K] \coloneqq \det \Sigma_{K,K}$ is a *principal minor* of Σ and $\Sigma[ij|K] \coloneqq \det \Sigma_{iK,jK}$ is an *almost-principal minor*.

If Σ is positive-definite, then $\Sigma[K] > 0$, and $\xi_i \perp\!\!\!\perp \xi_j \mid \xi_K$ holds if and only if $\Sigma[ij|K] = 0$.

Almost-principal minors

$$
\begin{aligned}
\Sigma[ij|] &= x_{ij} \\
\Sigma[ij|k] &= x_{ij}x_{kk} - x_{ik}x_{jk} \\
\Sigma[ij|kl] &= x_{ij}x_{kk}x_{ll} - x_{il}x_{jl}x_{kk} + x_{il}x_{jk}x_{kl} + x_{ik}x_{jl}x_{kl} - x_{ij}x_{kl}^2 - x_{ik}x_{jk}x_{ll} \\
\Sigma[ij|klm] &= x_{ij}x_{kk}x_{ll}x_{mm} + x_{im}x_{jm}x_{kl}^2 - x_{im}x_{jl}x_{kl}x_{km} - x_{il}x_{jm}x_{kl}x_{km} + x_{il}x_{jl}x_{km}^2 \\
&\quad - x_{im}x_{jm}x_{kk}x_{ll} + x_{im}x_{jk}x_{km}x_{ll} + x_{ik}x_{jm}x_{km}x_{ll} - x_{ij}x_{km}^2x_{ll} \\
&\quad + x_{im}x_{jl}x_{kk}x_{lm} + x_{il}x_{jm}x_{kk}x_{lm} - x_{im}x_{jk}x_{kl}x_{lm} - x_{ik}x_{jm}x_{kl}x_{lm} \\
&\quad - x_{il}x_{jk}x_{km}x_{lm} - x_{ik}x_{jl}x_{km}x_{lm} + 2x_{ij}x_{kl}x_{km}x_{lm} + x_{ik}x_{jk}x_{lm}^2 \\
&\quad - x_{ij}x_{kk}x_{lm}^2 - x_{il}x_{jl}x_{kk}x_{mm} + x_{il}x_{jk}x_{kl}x_{mm} + x_{ik}x_{jl}x_{kl}x_{mm} \\
&\quad - x_{ij}x_{kl}^2x_{mm} - x_{ik}x_{jk}x_{ll}x_{mm} \\
&\vdots
\end{aligned}
$$

Models and inference

Definition

A *CI constraint* is a CI statement $\xi_i \perp\!\!\!\perp \xi_j \mid \xi_K$ or its negation $\neg(\xi_i \perp\!\!\!\perp \xi_j \mid \xi_K)$. They are *algebraic conditions* on the entries of Σ, equivalent to vanishing or non-vanishing of the almost-principal minors $\Sigma[ij|K]$.

Definition

The *model* of a set of CI constraints is the set of all positive-definite matrices which satisfy the constraints.

Models and inference

Consider two sets of CI statements $\mathcal{L}$ and $\mathcal{M}$:

$$\bigwedge \mathcal{L} \Rightarrow \bigvee \mathcal{M}$$

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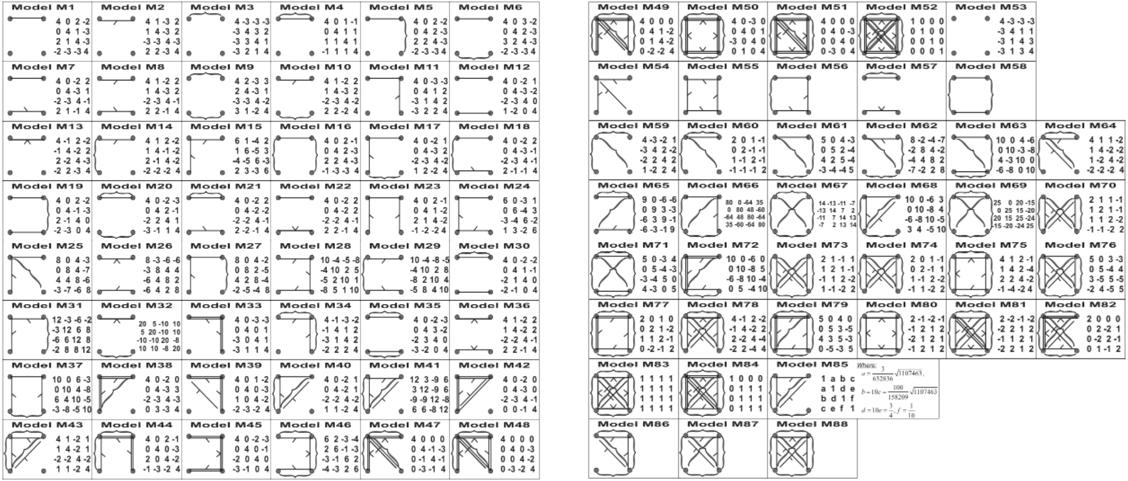
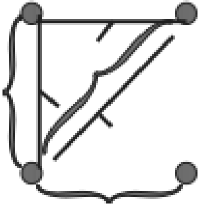
Reasoning about relevance statements in normally distributed random variables is the same as reasoning about the vanishing of very special kinds of determinants in the positive-definite matrices.



Petr Šimeček. Gaussian representation of independence models over four random variables. In *COMPSTAT conference*, 2006.

Model M85



$$\begin{pmatrix} 1 & a & b & c \\ a & 1 & d & e \\ b & d & 1 & f \\ c & e & f & 1 \end{pmatrix}$$

Where:

$$a = \frac{3}{632836}\sqrt{1107463},$$

$$b = 10c = \frac{100}{158209}\sqrt{1107463}$$

$$d = 10e = \frac{3}{4}, f = \frac{1}{10}$$

Šimeček's Question

Does every non-empty Gaussian CI model contain a rational point?

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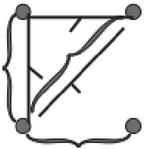

$$\begin{matrix} 1 & a & b & c \\ a & 1 & d & e \\ b & d & 1 & f \\ c & e & f & 1 \end{matrix}$$

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$$\begin{pmatrix} 1 & -1/17 & -49/51 & -7/17 \\ -1/17 & 1 & 1/3 & 1/7 \\ -49/51 & 1/3 & 1 & 3/7 \\ -7/17 & 1/7 & 3/7 & 1 \end{pmatrix}$$

Complexity bounds

Let $f_1, \dots, f_r \in \mathbb{Z}[t_1, \dots, t_k]$ be integer polynomials in finitely many variables. We consider a system of polynomial constraints "$f_i \bowtie 0$" where $\bowtie \in \{=, \neq, <, \leq, \geq, >\}$.

Theorem (Tarski's transfer principle)

If a polynomial system $\{f_i \bowtie 0\}$ has a solution over $\mathbb{R}$, then it has a solution in a finite real extension of $\mathbb{Q}$.

Theorem (Real Nullstellensatz)

A polynomial F vanishes on the semialgebraic set $\mathcal{K} = \{f_i \bowtie 0\}$ if and only if $F \in \sqrt[\mathbb{R}]{\mathcal{I}(f_i \bowtie 0)}$. The ideal $\mathcal{I}(f_i \bowtie 0)$, its real radical and the membership of F can be computed.

Keyword for this decision problem: "existential theory of the reals".

Main results

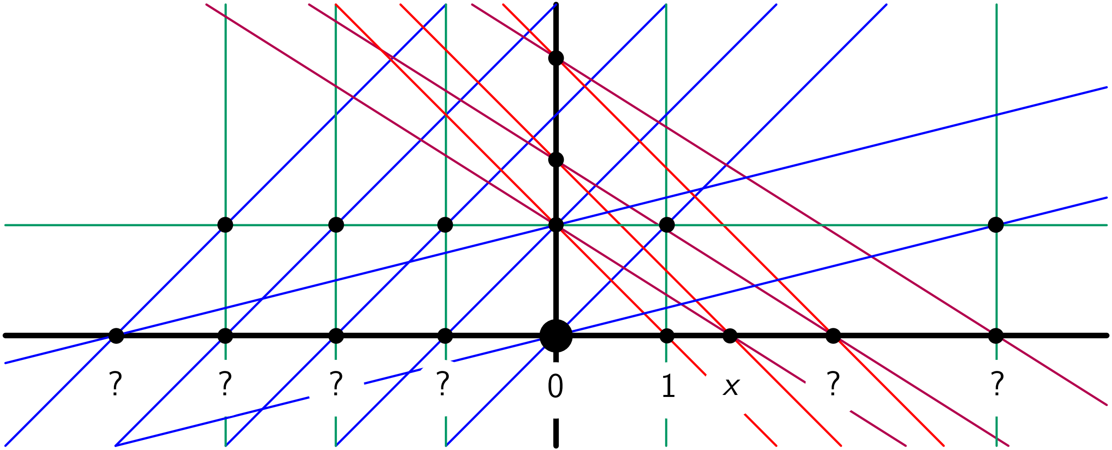
Theorem

Let $d \geq 1$ and $\mathbb{Q}^{(d)}$ the field generated by all real algebraic numbers of degree at most d. For every d there exists a non-empty Gaussian CI model which has no $\mathbb{Q}^{(d)}$-rational point.

Theorem

For every system of polynomials defining a semialgebraic set $\mathcal{K} = \{f_i \bowtie 0\}$ there exists a Gaussian CI model which is inhabited over $\mathbb{R}$ if and only if $\mathcal{K}$ is non-empty. Moreover, the description of this model is polynomially-sized in the description of $\mathcal{K}$.

Breakout riddle

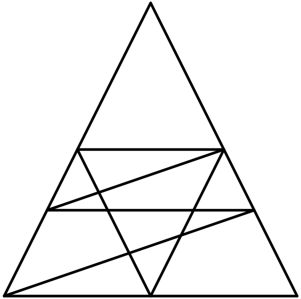


Algebra ⊆ Synthetic geometry

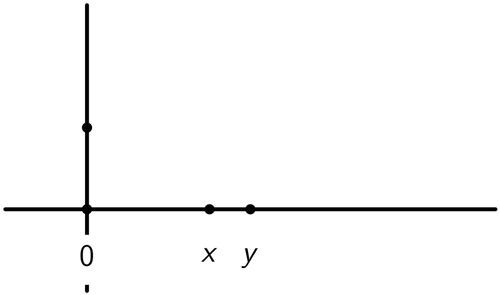
Point and line configuration for the equation $x^2 - 2 = 0$.

The configuration is specified by incidences between points and lines and also the parallelities of lines.

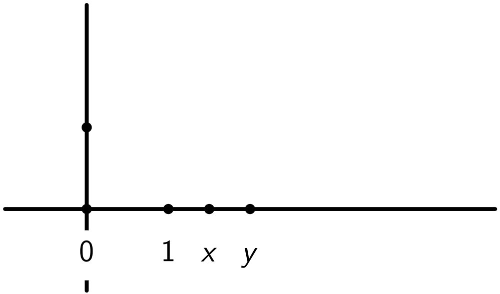
It is realizable over $\mathbb{Q}(\sqrt{2})$ but not over $\mathbb{Q}$.



Von Staudt constructions

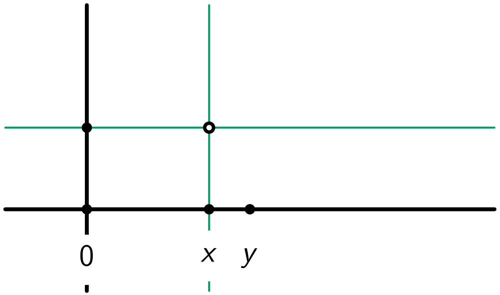


Addition



Multiplication

Von Staudt constructions



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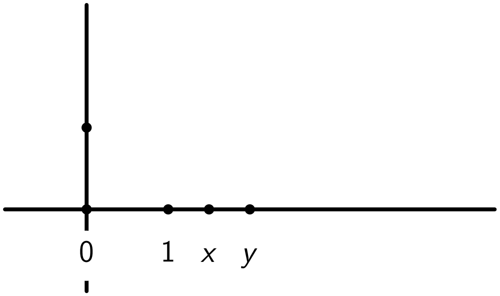

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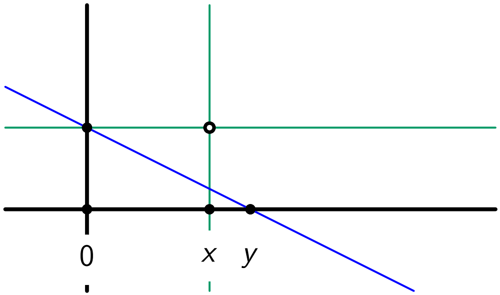

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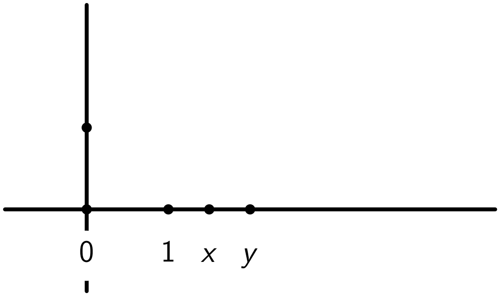

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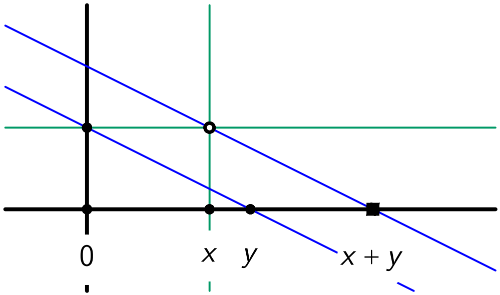

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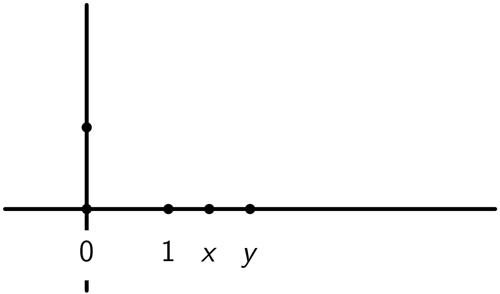

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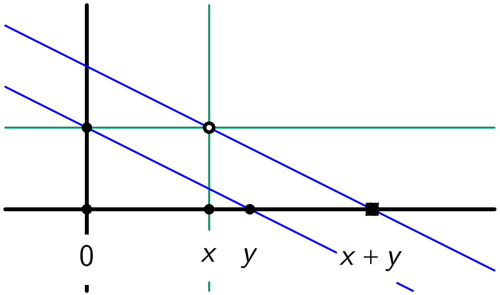

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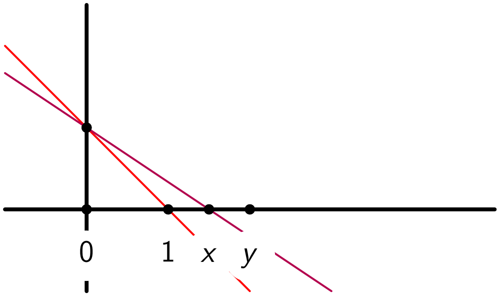

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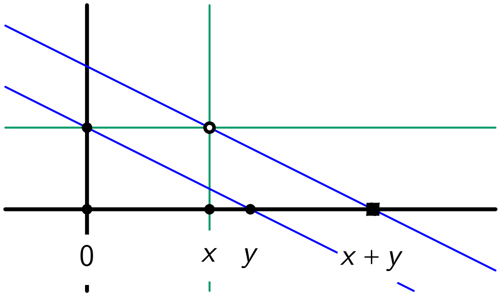

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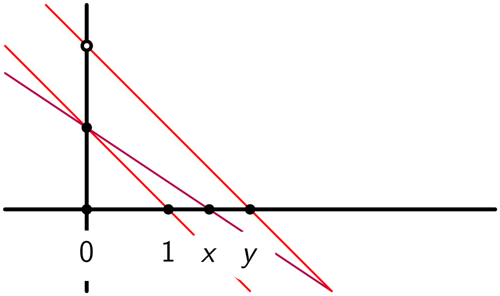

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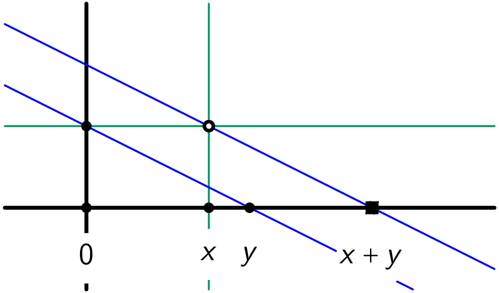

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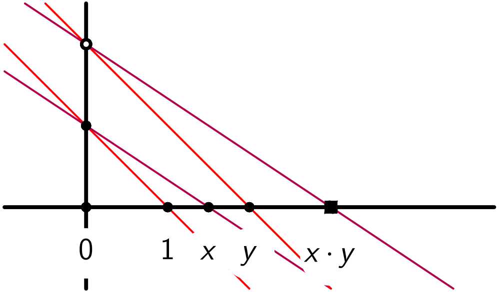

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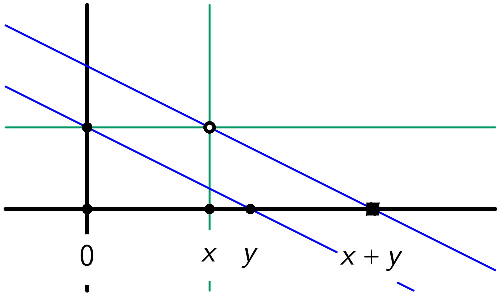

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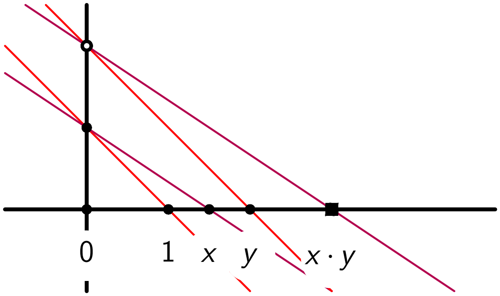
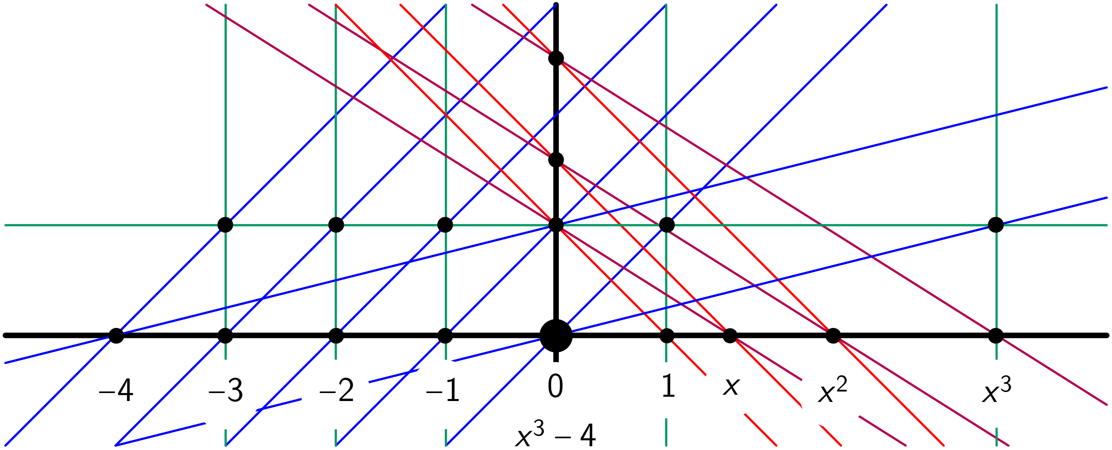


Multiplication

Lemma

The evaluation of integer polynomials can be encoded with incidence geometry.

The cube root of 4



Incidence relations as conditional independence

Suppose $E = \{x, y, z\}$ and Σ_E is the identity matrix.

$$\Sigma[ij|E] = \Sigma[E]\left(\Sigma_{ij} - \Sigma_{i,E}\Sigma_E^{-1}\Sigma_{E,j}\right) \qquad (\measuredangle)$$

$$\left.\begin{array}{l} \Sigma[ij|E] = 0 \Leftrightarrow \Sigma_{ij} = \langle \Sigma_{i,E}, \Sigma_{j,E} \rangle \\ \Sigma[ij|] = 0 \Leftrightarrow \Sigma_{ij} = 0 \end{array}\right\} \Leftrightarrow \Sigma_{i,E} \perp \Sigma_{j,E}$$

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$p = \Sigma_{i,E} = [p_x : p_y : p_z]$ and $\ell = \Sigma_{j,E} = [\ell_x : \ell_y : \ell_z]$ are the *homogeneous coordinates* of a point and a line in the projective plane with $p \in \ell^\perp$.

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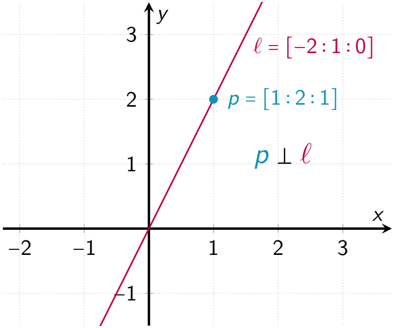

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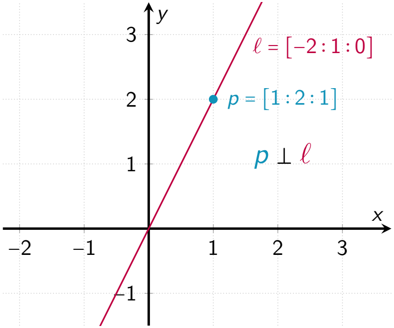
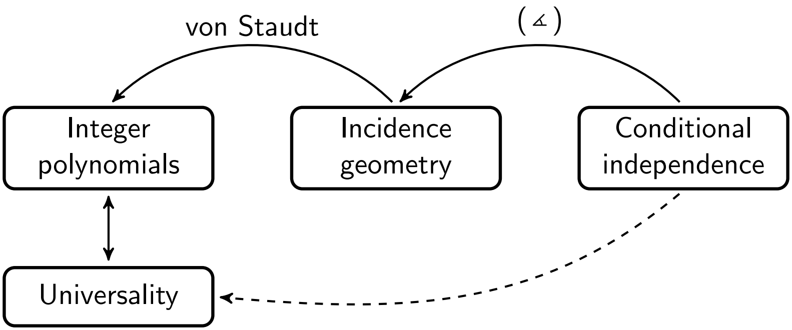


Lemma

Incidence geometry can be encoded in CI constraints.

Polynomial evaluation as conditional independence

$$
\begin{array}{c}
\\ \mathrm{p}_1 \\ \vdots \\ \mathrm{p}_n \\ \mathrm{l}_1 \\ \vdots \\ \mathrm{l}_m \\ \mathrm{x} \\ \mathrm{y} \\ \mathrm{z}
\end{array}
\begin{array}{c}
\begin{array}{ccccccccc}
\mathrm{p}_1 & \cdots & \mathrm{p}_n & \mathrm{l}_1 & \cdots & \mathrm{l}_m & \mathrm{x} & \mathrm{y} & \mathrm{z}
\end{array} \\
\left(\begin{array}{ccc|ccc|ccc}
p_1^* & & \langle p, p' \rangle & & & & p_1^x & p_1^y & p_1^z \\
& \ddots & & & \langle p, \ell \rangle & & & \vdots & \\
\langle p', p \rangle & & p_n^* & & & & p_n^x & p_n^y & p_n^z \\
\hline
& & & \ell_1^* & & \langle \ell, \ell' \rangle & \ell_1^x & \ell_1^y & \ell_1^z \\
& \langle \ell, p \rangle & & & \ddots & & & \vdots & \\
& & & \langle \ell', \ell \rangle & & \ell_m^* & \ell_m^x & \ell_m^y & \ell_m^z \\
\hline
p_1^x & & p_n^x & \ell_1^x & & \ell_m^x & 1 & 0 & 0 \\
p_1^y & \cdots & p_n^y & \ell_1^y & \cdots & \ell_m^y & 0 & 1 & 0 \\
p_1^z & & p_n^z & \ell_1^z & & \ell_m^z & 0 & 0 & 1
\end{array}\right)
\end{array}
$$



Theorem

To every polynomial system $\{f_i \bowtie 0\}$ *there is a set of CI constraints which has a model over a field* $\mathbb{K}/\mathbb{Q}$ *if and only if the polynomial system has a solution in* $\mathbb{K}$.

QED

Theorem

Let $d \geq 1$ and $\mathbb{Q}^{(d)}$ the field generated by all real algebraic numbers of degree at most d. For every d there exists a non-empty Gaussian CI model which has no $\mathbb{Q}^{(d)}$-rational point.

Theorem

For every system of polynomials defining a semialgebraic set $\mathcal{K} = \{f_i \bowtie 0\}$ there exists a Gaussian CI model which is inhabited over $\mathbb{R}$ if and only if $\mathcal{K}$ is non-empty. Moreover, the description of this model is polynomially-sized in the description of $\mathcal{K}$.

Question: What is the smallest n (≥ 5) for which there is an n-variate Gaussian CI model without rational point?

Tobias Boege.
Incidence geometry in the projective plane via almost-principal minors of symmetric matrices, 2021.
arXiv:2103.02589.

Jürgen Bokowski and Bernd Sturmfels.
Computational synthetic geometry, volume 1355 of *Lecture Notes in Mathematics*.
Springer, 1989.

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Perspectives on projective geometry. A guided tour through real and complex geometry.
Springer, 2011.

Petr Šimeček.
Gaussian representation of independence models over four random variables.
In *COMPSTAT conference*, 2006.

$\bowtie \in \{=\}$ suffices

In the real numbers, the solvability of a system of equations is just as hard as equations, inequations and inequalities if we introduce a new variable y:

- $f(x) \neq 0 \;\Leftrightarrow\; \exists y : yf(x) - 1 = 0$ ($f(x)$ has a multiplicative inverse)
- $f(x) \geq 0 \;\Leftrightarrow\; \exists y : f(x) - y^2 = 0$ ($f(x)$ is a square, i.e. non-negative)
- $f(x) > 0 \;\Leftrightarrow\; \exists y : y^2 f(x) - 1 = 0$ ($f(x)$ has an inverse which is a square)